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Non parametric Fractional Cointegration Analysis

by

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ABSTRACT

This paper provides a theoretical fractional cointegration analysis in a nonparametric framework. We solve a generalized eigenvalues problem. To this end, a couple of random matrices are constructed taking into account the stationarity properties of the differencesof a fractional p-variate integrated process. These difference orders are assumed to vary in a continuous and discrete range. The random matrices are defined by some weight functions. Asymptotic behaviors of these random matrices are obtained by stating some conditions on the weight functions, and by using Bierens (1997) and Andersen et al.(1983) results. In this way, a nonparametric analysis is provided. Moving from the solution of the generalized eigenvalue problem, a fractional nonparametric VAR model for cointegration is
also presented.

Keywords: Fractional integrated process, Nonparametric methods, Cointegration, Asymptotic distribution, Generalized eigenvalues problem.

JEL Classification: C14, C22, C65.

NON-TECHNICAL SUMMARY

The concept of cointegration has been introduced by Granger (1981) and analyzed by Engle and Granger (1987). Most of the analyses have mainly considered the CI(1,1) cointegration case, in which two or more I(1) variables give rise to I(0) linear combinations and the long run relationships are derived with little or no restrictions on the short run dynamics. In order to avoid the knife-edge I(1)/I(0) distinction and to allow for potential slow adjustments towards long run equilibria, fractional cointegration approaches have been proposed.

In this paper new theoretical analysis on cointegration is proposed. The contribution of this work to the literature on cointegration is as follows: First, a nonparametric approach to solve a generalized eigenvalue problem for fractional integrated process is given. Second, a fractional nonparametric VAR model to show the usefulness of our theoretical analysis is presented.

The eigenvalue problem is solved by considering the asymptotic behavior of two random matrices. Such matrices are constructed by taking into account the stationarity properties of the differences of a fractional p-variate integrated process. These difference orders are assumed to vary in a continuous and discrete range. The continuous case is general, since it consider the whole set of information. However, to let our analysis be useful in economic applications, the discrete case is also provided in such a way that the most part of the differences orders are included. The asymptotic convergence results give nonparametric analysis.

UN’ANALISI DI COINTEGRAZIONE FRAZIONATA

SINTESI

In questo lavoro si propone un approccio teorico allo studio della cointegrazione in un contesto frazionato. Il problema generalizzato degli autovalori viene risolto con una tecnica nonparametrica. Per risolvere tale problema, si costruiscono due matrici casuali, distinguendo la parte non-stazionaria e stazionaria delle differenze del processo frazionato. L’ordine delle differenze varia nel campo continuo e discreto. Il caso discreto è considerato perchè particolarmente utile per le applicazioni economiche. Le due matrici casuali sono inoltre definite sulla base di alcune funzioni peso. Delle matrici casuali si ottengono le distribuzioni asintotiche definendo alcune condizioni sulle funzioni peso. L’analisi non-parametrica è ottenuta con la convergenza asintotica. Una volta risolto il problema generalizzato degli autovalori, si presenta un modello VAR frazionato.

Parole chiave: Processi integrati frazionati, Metodi nonparametrici, Cointegrazione, Distribuzioni asintotiche, Problema generalizzato degli autovalori.

Classificazione JEL: C14, C22, C65.

CONTENTS

1 INTRODUCTION[1]

The concept of cointegration has been introduced by Granger (1981) and analyzed by Engle and Granger (1987). Most of the analyses have mainly considered the CI(1,1) cointegration case, in which two or more I(1) variables give rise to I(0) linear combinations and the long run relationships are derived with little or no restrictions on the short run dynamics. In order to avoid the knife-edge I(1)/I(0) distinction and to allow for potential slow adjustments towards long run equilibria, fractional cointegration approaches have been proposed. Consider a two-dimensional process $(X_t; Y_t)$ such that both variables are I(d) processes. We say that X_t and Y_t are fractionally cointegrated if there exists a linear combination $U_t = Y_t - BX_t$ such that U_t is $I(d_U)$, with $d_U < d$. Fractional cointegration is a generalization of standard cointegration, where d and d_U are 1 and 0, respectively. Parametric and semiparametric fractional cointegration models have focused on the reduction of the memory parameter from $d \geq \frac{1}{2}$ to $d_U < \frac{1}{2}$, since cointegration is commonly thought if as a stationary relationship between stationary, but cases in which the differencing parameter is less than $\frac{1}{2}$ are also discussed, in particular in the context of financial time series analysis. A partial list of works includes Jeganathan (1999), Breitung and Hassler (2002), Davidson (2002), Robinson and Yajima (2002), Robinson and Hualde (2003), Nielsen (2004), Marmol and Velasco (2004). While Jeganathan (1999), Phillips and Kim (2001), Breitung and Hassler (2002), Davidson (2002) and Robinson and Hualde (2003) and Marmol *et al.* (2002) developed parametric models, Robinson and Yajima (2002), Nielsen (2004), Marmol and Velasco (2004) and Robinson and Iacone (2005) worked in a semiparametric context.

In this paper new theoretical analysis on cointegration is proposed. The contribution of this work to the literature on cointegration is as follows: First, a nonparametric approach to solve a generalized eigenvalue problem for fractional integrated process is given. Second, a fractional nonparametric VAR model to

[1]The authors would like to thank Massimo Franchi and Paolo Paruolo for helpful comments and suggestions.

show the usefulness of our theoretical analysis is presented.

The eigenvalue problem is defined by considering a combination of two random matrices. Such matrices are constructed by taking into account the stationarity properties of the differences of a fractional p-variate integrated process. These difference orders are assumed to vary in a continuous and discrete range. The continuous case is general, since it consider the whole set of information. However, to let our analysis be useful in economic applications, a discretization of the continuous case is provided. The set of the rational number $\mathbf{Q}$ is considered, since $\mathbf{Q}$ is dense in $\mathbf{R}$, and then the most part of the differences orders are included. The random matrices are defined by some weight functions. Asymptotic behaviors of these random matrices are obtained and nonparametric analysis is provided. Moving from the solution of the generalized eigenvalue problem, a fractional nonparametric VAR model for cointegration is presented.

The paper is organized as follows. Section 2 presents the data generating process. In Section 3 the continuous case is studied. Section 4 presents the discrete case. In section 5 nonparametric fractional VAR model is proposed. Section 6 concludes.

2 DATA GENERATING PROCESS

In this section we describe the data generating process.

The data generating process Y_t is assumed to be a fractional non explosive integrated process of order d satisfying the following definition.

Defintion 2.1 *Given $p \in \mathbf{N}$, a p-variate time series $\{Y_t\}$ is a fractional integrated process with fractional degree of integration $-1/2 < d < 1$ if*

$$Y_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j} \qquad \text{with } c_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}, \tag{1}$$

where $\{\epsilon_t\}_{t>0}$ is an i.i.d. p-variate vector sequence with zero mean. We denote $Y_t \sim I(d)$.

Following Robinson (2003), we have:

1. if $-1/2 < d \leq 1/2$, then Y_t is stationary;

2. if $1/2 < d < 1$, then Y_t is nonstationary, non explosive and nonpersistent.

Assumption 2.1 *There exists a p-squared matrix of lag polynomials in the lag operator L such that*

$$\epsilon_t = \sum_{j=0}^{\infty} C_j v_{t-j} =: C(L)v_t, \quad t = 1, \ldots, n, \tag{2}$$

where v_t is a p-variate stationary white noise process. Now we state some hypotheses on $C(L)$.

Assumption 2.2 *The process ϵ_t can be written as in (2), where v_t are i.i.d. zero-mean p-variate gaussian variables with variance equals to the identity matrix of order p, I_p, and there exist $C_1(L)$ and $C_2(L)$ p-squared matrices of lag polynomials in the lag operator L such that all the roots of* $\det C_1(L)$ *are outside the complex unit circle and $C(L) = C_1(L)^{-1}C_2(L)$.* The lag polynomial $C(L) - C(1)$ attains value zero at $L = 1$ with fractional algebraic multiplicity equals to d. Thus, there exists a lag fractional polynomial

$$D(L) = \sum_{k=0}^{\infty} D_k L^{\zeta_k}, \qquad D_k,\, \zeta_k \in \mathbf{R},\ \forall\, k = 1, \ldots, +\infty,$$

such that $C(L) - C(1) = (1-L)^d D(L)$ and ζ_k is increasing.

Therefore, we can write

$$\epsilon_t = C(L)v_t = C(1)v_t + [C(L) - C(1)]v_t = C(1)v_t + D(L)(1-L)^d v_t. \tag{3}$$

Let us define $w_t := D(L)v_t$. Then, substituting w_t into (3), we get

$$\epsilon_t = C(1)v_t + (1-L)^d w_t. \tag{4}$$

(4) implies that, given $Y_t \sim I(d)$, we can write recursively

$$\Delta^{d-1}Y_t = \Delta^{d-1}Y_{t-1} + \epsilon_t = \Delta^{d-1}Y_0 + (1-L)w_t - w_0 + C(1)\sum_{j=1}^{t} v_j. \tag{5}$$

If $\text{rank}(C(1)) = p - r < p$, then the process $\Delta^{d-1}Y_t$ is cointegrated with r linear independent cointegrating vectors. Since $d < 1$, if $d - 1 < \alpha < d + 1/2$, then $\Delta^{\alpha}Y_t$ is cointegrated with r cointegrating vectors $\gamma_1, \ldots, \gamma_r$. In fact

$$Y_t \sim I(d) \Rightarrow \Delta^{\alpha}Y_t \sim I(d-\alpha), \qquad -1/2 < d - \alpha < 1. \tag{6}$$

Assumption 2.3 *Let us consider R_r the matrix of the eigenvectors of $C(1)C(1)^T$ corresponding to the r zero eigenvalues. Then the matrix $R_r^T D(1)D(1)^T R_r$ is nonsingular.*

Assumption 2.3 implies that Y_t cannot be integrated of order $\bar{d}$, with $\bar{d} > d$. In fact, if there exists $\bar{d} > d$ such that $Y_t \sim I(\bar{d})$, then the lag polynomial $D(L)$ admits a unit root with algebraic multiplicity $\bar{d} - d$, and so $D(1)$ is singular. Therefore $R_r^T D(1)D(1)^T R_r$ is singular, and Assumption 2.2 does not hold.

3 THE GENERALIZED EIGENVALUES PROBLEM: THE CONTINUOUS CASE

The aim of this section is to construct a couple of random matrices, in order to address the solution of the generalized eigenvalue problem.

We want to emphasize that such random matrices take into account the stationary and nonstationary part of the data generating process. To this end, we rely on the α-th differences of Y_t that can be stationary or nonstationary processes, depending on the choice of α. The relationship between the difference orders and the related process can be described as follows:

- if $d - 1 < \alpha < d - 1/2$, then $\Delta^\alpha Y_t$ is nonstationary;
- if $d - 1/2 < \alpha < d + 1/2$, then $\Delta^\alpha Y_t$ is stationary.

In this first part of the theoretical nonparametric cointegration framework, the entire set of the admissible differences of Y_t is considered. Fixed $\alpha \in (d - 1, d + 1/2)$, the α-th difference of the process Y_t is opportunely weighted by some functions depending on α. Then, all these terms are aggregated by integrating on α.

The random matrices are assumed to be dependent on an integer number $m \geq p$. Let us fix $k = 1, \ldots, m$, and define the functions

$$F_k : [0, 1] \to \mathbf{R}; \tag{7}$$

$$G_{k,\alpha} : [0,1] \to \mathbf{R}, \qquad \alpha \in (d-1, d-1/2);$$

$$H_{k,\alpha} : [0,1] \to \mathbf{R}, \qquad \alpha \in (d-1/2, d+1/2).$$

Moreover, we consider a couple of sequences:

$$\{\phi_1(n,\alpha)\} \subseteq \mathbf{R}, \qquad \alpha \in (d-1, d-1/2);$$

$$\{\phi_2(n,\alpha)\} \subseteq \mathbf{R}, \qquad \alpha \in (d-1/2, d+1/2).$$

By using the previous definitions of functions and sequences, the random matrices are constructed. They are, respectively,

$$A_m := \sum_{k=1}^{m} a_{n,k} a_{n,k}^T; \tag{8}$$

$$B_m := \sum_{k=1}^{m} b_{n,k} b_{n,k}^T, \tag{9}$$

where

$$a_{n,k} := \frac{M_n^{nonst}/\sqrt{n}}{\sqrt{\int\int F_k(x)F_k(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y}}; \tag{10}$$

$$b_{n,k} := \frac{\sqrt{n}M_n^{st}}{\sqrt{\int F_k(x)^2\mathrm{d}x}}, \tag{11}$$

and

$$M_n^{nonst} = \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta^{d-1}Y_t + \int_{d-1}^{d-1/2}\Big[\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha; \tag{12}$$

$$M_n^{st} = \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta^{d}Y_t + \int_{d-1/2}^{d+1/2}\Big[\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha, \tag{13}$$

The main result of this work is obtained by an asymptotic analysis of a particular combination of the random matrices. These random matrices are defined on the basis of the weight functions F's, G's and H's is provided. Two definitions are proposed in order to show that these weight functions belong to three functional classes.

Definition 3.1 *Let us fix $m \in \mathbf{N}$, $k = 1, \dots m$.*

(i) There exists a function $\theta_1 : (d-1, d-1/2) \to \mathbf{R}$ and $\phi_1 : \mathbf{N} \times (d-1, d-1/2) \to \mathbf{R}$ such that

$$\alpha \mapsto \theta_1(\alpha), \qquad \theta_1 \in L^1(d-1, d-1/2)$$

and

$$\Big| \sqrt{n}\phi_1(n, \alpha) \sum_{t=1}^{n} G_{k,\alpha}(t/n) \Big| \leq \theta_1(\alpha), \qquad \forall\, \alpha \in (-\infty, d-1/2).$$

(ii) For each $\alpha \in (-\infty, d-1/2)$, it results

$$\lim_{n \to +\infty} \sqrt{n}\phi_1(n, \alpha) \sum_{t=1}^{n} G_{k,\alpha}(t/n) = 0; \tag{14}$$

(iii) There exists a function $\theta_2 : (d-1/2, d+1/2) \to \mathbf{R}$ and $\phi_2 : \mathbf{N} \times (d-1/2, d+1/2) \to \mathbf{R}$ such that

$$\alpha \mapsto \theta_2(\alpha), \qquad \theta_2 \in L^1(d-1/2, d+1/2)$$

and

$$\Big| n\phi_2(n, \alpha) \sum_{t=1}^{n} H_{k,\alpha}(t/n) \Big| \leq \theta_2(\alpha), \qquad \forall\, \alpha \in (d-1/2, d+1/2).$$

(iv) For each $\alpha \in (d-1/2, d+1/2)$, it results

$$\lim_{n \to +\infty} n\phi_2(n, \alpha) \sum_{t=1}^{n} H_{k,\alpha}(t/n) = 0; \tag{15}$$

The functional classes $\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are

$$\mathcal{G}_{m,\alpha} := \Big\{ G_{k,\alpha} : [0,1] \to \mathbf{R} \,|\, (i),\, (ii) \text{ hold} \Big\}. \tag{16}$$

$$\mathcal{H}_{m,\alpha} := \Big\{ H_{k,\alpha} : [0,1] \to \mathbf{R},\, |\, (iii),\, (iv) \text{ hold} \Big\}. \tag{17}$$

Definition 3.2 Consider the following conditions:

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} F_k(t/n) = o(1); \tag{18}$$

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} tF_k(t/n) = o(1); \tag{19}$$

$$\int\int F_i(x)F_j(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y = 0, \qquad i \neq j; \tag{20}$$

$$\int F_i(x)\int_0^x F_j(y)\mathrm{d}x\mathrm{d}y = 0, \qquad i \neq j; \tag{21}$$

$$\int F_i(x)F_j(x)\mathrm{d}x = 0, \qquad i \neq j. \tag{22}$$

The functional class $\mathcal{F}_m$ is

$$\mathcal{F}_m := \Big\{ F_k : [0,1] \to \mathbf{R},\, F_k \in C^1[0,1] \,|\, (18) \,-\, (22) \text{ hold} \Big\}. \tag{23}$$

Bierens (1997) shows that the functional class $\mathcal{F}_m$ is not empty. He pointed out that, if we define

$$\bar{F}_k : \mathbf{R} \to \mathbf{R}$$

such that

$$\bar{F}_k(x) = cos(2k\pi x), \tag{24}$$

and taking the restriction

$$F_k := \bar{F}_k|_{[0,1]},$$

then $F_k \in \mathcal{F}_m$.

Moreover, $\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are not empty, and they contain a huge number of elements. Therefore, it is not restrictive to assume that the weights G's and H's belong to these spaces. Some properties of $\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are showed, in order to evidence the big cardinality of these spaces.

Proposition 3.1 *$\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are closed with respect to the linear combination.*

Proof We provide the proof only for the functional space $\mathcal{G}_{m,\alpha}$, being the one for $\mathcal{H}_{m,\alpha}$ analogous.

Given $k = 1, \ldots, m$ and $\alpha \in (-\infty, d - 1/2)$, let us consider

$$G^j_{k,\alpha} : [0,1] \to \mathbf{R}, \qquad j = 1, \ldots, N,\ N \in \mathbf{N}$$

such that $G^j_{k,\alpha} \in \mathcal{G}_{m,\alpha}$.
Define

$$G_{k,\alpha} := \sum_{j=1}^{N} q_j G^j_{k,\alpha}, \qquad q_j \in \mathbf{R},\ \forall\, j = 1, \ldots, N.$$

Conditions (i) and (ii) of Definition 3.1 can be rewritten by indexing with j the sequence ϕ_1 and the function θ_1. We rewrite them, for sake of completeness. Fix $j = 1, \ldots, N$, where $N \in \mathbf{N}$. Then

(i) There exists a function $\theta^j_1 : (-\infty, d - 1/2) \to \mathbf{R}$ and $\phi^j_1 : \mathbf{N} \times (-\infty, d - 1/2) \to \mathbf{R}$ such that

$$\alpha \mapsto \theta^j_1(\alpha), \qquad \theta_1 \in L^1(-\infty, d - 1/2)$$

and

$$\left| \sqrt{n}\phi^j_1(n,\alpha) \sum_{t=1}^{n} G^j_{k,\alpha}(t/n) \right| \leq \theta^j_1(\alpha), \qquad \forall\, \alpha \in (-\infty, d - 1/2).$$

(ii) For each $\alpha \in (-\infty, d - 1/2)$, it results

$$\lim_{n \to +\infty} \sqrt{n}\phi^j_1(n,\alpha) \sum_{t=1}^{n} G^j_{k,\alpha}(t/n) = 0; \tag{25}$$

The condition (i) is fulfilled. In fact, by choosing ϕ_1 such that

$$\phi_1(n,\alpha) = o(\phi^j_1(n,\alpha)), \qquad \forall\, j = 1, \ldots, N, \text{ as } n \to +\infty,$$

then

$$\lim_{n \to +\infty} \sqrt{n}\phi_1(n,\alpha) \sum_{t=1}^{n} G_{k,\alpha}(t/n) = \lim_{n \to +\infty} \sqrt{n}\phi_1(n,\alpha) \sum_{t=1}^{n} \Big[\sum_{j=1}^{N} q_j G^j_{k,\alpha}(t/n) \Big] =$$

$$= \sum_{j=1}^{N} q_j \Big[\lim_{n \to +\infty} \sqrt{n}\phi_1(n,\alpha) \sum_{t=1}^{n} G^j_{k,\alpha}(t/n) \Big] = 0.$$

Furthermore, by using ϕ_1 as above, it results

$$\left| \sqrt{n}\phi_1(n,\alpha) \sum_{t=1}^{n} G_{k,\alpha}(t/n) \right| = \left| \sqrt{n}\phi_1(n,\alpha) \sum_{t=1}^{n} \Big[\sum_{j=1}^{N} q_j G^j_{k,\alpha}(t/n) \Big] \right| \leq$$

$$\leq \sum_{j=1}^{N} |q_j| \Big| \sqrt{n}\phi_1(n,\alpha) \sum_{t=1}^{n} G^j_{k,\alpha}(t/n) \Big| \leq \sum_{j=1}^{N} |q_j| \theta_1^j(\alpha).$$

Since

$$\sum_{j=1}^{N} |q_j| \theta_1^j(\alpha) \in L^1(-\infty, d-1/2),$$

condition (ii) holds.

As a consequence of the previous result, an interesting topological property of $\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ can be obtained.

Corollary 3.1 *$\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are convex sets.*

Proof. We provide only the proof for the functional space $\mathcal{G}_{m,\alpha}$.

For $k = 1, \dots, m$ and $\alpha \in (-\infty, d-1/2)$, we define a couple of functions

$$G^j_{k,\alpha} : [0,1] \to \mathbf{R}, \qquad j = 1, 2.$$

such that $G^j_{k,\alpha} \in \mathcal{G}_{m,\alpha}$.

Define $q_1, q_2 \in [0,1]$ such that $q_1 + q_2 = 1$, and the convex linear combination function

$$G_{k,\alpha} := q_1 G^1_{k,\alpha} + q_2 G^2_{k,\alpha}.$$

Since Proposition 3.1 implies $G_{k,\alpha} \in \mathcal{G}_{m,\alpha}$, we have the thesis. ■

We want now to show a sufficient condition to characterize $\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$.

Theorem 3.2 *Fix $\alpha \in (-\infty, d+1/2)$ and $k = 1, \dots, m$. Define $\varrho_{k,\alpha} : [0,1] \to \mathbf{R}$, and assume that there exists $M > 0$ such that*

$$|\varrho_{k,\alpha}(x)| \leq M, \qquad \forall\, x \in [0,1].$$

Then:

- *$\varrho_{k,\alpha}$ belongs to $\mathcal{G}_{m,\alpha}$ if $\alpha \in (-\infty, d-1/2)$;*
- *$\varrho_{k,\alpha}$ belongs to $\mathcal{H}_{m,\alpha}$ if $\alpha \in (d-1/2, d+1/2)$.*

Proof. For sake of simplicity, we denote ϱ as H, when $\alpha \in (d-1/2, d+1/2)$, and as G, when $\alpha \in (-\infty, d-1/2)$.

Standard analysis provides that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{t=1}^{n} G_{k,\alpha}(t/n) = \int_0^1 G_{k,\alpha}(x) \mathrm{d}x. \tag{26}$$

Fixed $\alpha \in (-\infty, d-1/2)$, we define a sequence $\{\psi_1(n,\alpha)\}_{n\in\mathbf{N}}$ such that

$$\phi_1(n,\alpha) = \frac{1}{n^{3/2}} \cdot \psi_1(n,\alpha), \tag{27}$$

$$\lim_{n\to+\infty} \psi_1(n,\alpha) = 0. \tag{28}$$

Moreover, fixed $n \in \mathbf{N}$, we assume that $\psi_1 \in L^1(-\infty, d-1/2)$.

By (27), we have

$$\Big|\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| = \Big|\psi_1(n,\alpha)\frac{1}{n}\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| \leq$$

$$\leq |\psi_1(n,\alpha)|\Big|\frac{1}{n}\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| \leq |\psi_1(n,\alpha)|\Big|\frac{1}{n}\cdot n\cdot M\Big| = M|\psi_1(n,\alpha)|.$$

By assuming $\theta_1(\alpha) = |\psi_1(n,\alpha)|$, condition (i) of Definition 3.1 holds.

Furthermore, it results

$$0 \leq \Big|\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| \leq M|\psi_1(n,\alpha)|.$$

Thus, by (28) and by using a comparison principle, we get

$$\lim_{n\to+\infty} \sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n) = 0.$$

Then, (ii) of Definition 3.1 holds, and $G_{k,\alpha} \in \mathcal{G}_{m,\alpha}$.

Now, fixed $\alpha \in (d-1/2, d+1/2)$, we define a sequence $\{\psi_2(n,\alpha)\}_{n\in\mathbf{N}}$ such that

$$\phi_2(n,\alpha) = \frac{1}{n^2} \cdot \psi_2(n,\alpha), \tag{29}$$

$$\lim_{n\to+\infty} \psi_2(n,\alpha) = 0. \tag{30}$$

Moreover, we assume that $|\psi_2| \in L^1(d-1/2, d+1/2)$.

By (29), some algebra gives

$$\Big|n\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Big| = \Big|\psi_2(n,\alpha)\frac{1}{n}\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Big| \leq$$

$$\leq |\psi_2(n,\alpha)|\Big|\frac{1}{n}\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Big| \leq |\psi_2(n,\alpha)|\Big|\frac{1}{n}\cdot n\cdot M\Big| = M|\psi_2(n,\alpha)|.$$

By assuming $\theta_2 = |\psi_2|$, condition (iii) of Definition 3.1 holds.
Furthermore, as showed for $G_{k,\alpha}$, (30) and a comparison principle give

$$\lim_{n\to+\infty} n\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n) = 0.$$

Then, (iv) of Definition 3.1 is satisfied, and so $H_{k,\alpha} \in \mathcal{H}_{m,\alpha}$. ■

3.1 Asymptotic results

In this section the main asymptotic result is presented in order to provide a nonparametric analysis of the generalized eigenvalue problem. At this aim, two random vectors dependent on the weight functions F's are defined as follows:

$$\Psi_k := \frac{\int_0^1 F_k(x)W(x)\mathrm{d}x}{\sqrt{\int_0^1\int_0^1 F_k(x)F_k(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y}},$$

$$\Phi_k := \frac{F_k(1)W(1) - \int_0^1 f_k(x)W(x)\mathrm{d}x}{\int_0^1 F_k(x)^2\mathrm{d}x},$$

where f_k is the derivative of F_k.
Moreover, we introduce the following p-variate standard normally distributed random vectors:

$$\Psi_k^* := \left(R_{p-r}^T C(1)C(1)^T R_{p-r}\right)^{\frac{1}{2}} R_{p-r}^T C(1)\Psi_k,$$

$$\Phi_k^* := \left(R_{p-r}^T C(1)C(1)^T R_{p-r}\right)^{\frac{1}{2}} R_{p-r}^T C(1)\Phi_k,$$

$$\Phi_k^{**} := (R_r^T D(1)D(1)^T R_r)^{-\frac{1}{2}} R_r^T D(1)\Phi_k,$$

and we construct the matrix $V_{r,m}$ as

$$V_{r,m} := (R_r^T D(1)D(1)^T R_r)^{\frac{1}{2}} V_{r,m}^* (R_r^T D(1)D(1)^T R_r)^{\frac{1}{2}},$$

with

$$V_{r,m}^* = \Big(\sum_{k=1}^m \gamma_k^2 \Phi_k^{**}\Phi_k^{**T}\Big) - \Big(\sum_{k=1}^m \gamma_k \Phi_k^{**}\Psi_k^{*T}\Big)\Big(\sum_{k=1}^m \Psi_k^*\Psi_k^{*T}\Big)^{-1}\Big(\sum_{k=1}^m \gamma_k \Psi_k^*\Phi_k^{**T}\Big),$$

where

$$\gamma_k = \frac{\sqrt{\int_0^1 F_k^2(x)\mathrm{d}x}}{\sqrt{\int_0^1\int_0^1 F_k(x)F_k(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y}}.$$

The following result summarizes the eigenvalue problem and provide a nonparametric solution for it.

3.3 Theorem *Assume that* $F_k \in \mathcal{F}_m$*,* $G_{k,\alpha} \in \mathcal{G}_{m,\alpha}$ *and* $H_{k,\alpha} \in \mathcal{H}_{m,\alpha}$*.*
If Assumptions 2.1, 2.2 and 2.3 are true, then:

(I) suppose that $\hat{\lambda}_{1,m} \geq \ldots \geq \hat{\lambda}_{p,m}$ *are the ordered solutions of the generalized eigenvalue problem*

$$\det\Big[A_m - \lambda(B_m + n^{-2}A_m^{-1})\Big] = 0, \tag{31}$$

and $\lambda_{1,m} \geq \ldots \geq \lambda_{p-r,m}$ *the ordered solutions of*

$$\det\Big[\sum_{k=1}^{m} \Psi_k^* \Psi_k^{*T} - \lambda \sum_{k=1}^{m} \Phi_k^* \Phi_k^{*T}\Big] = 0. \tag{32}$$

Then we have the following convergence in distribution

$$(\hat{\lambda}_{1,m}, \ldots, \hat{\lambda}_{p,m}) \to (\lambda_{1,m}, \ldots, \lambda_{p-r,m}, 0, \ldots, 0);$$

(II) let us consider $\lambda_{1,m}^* \geq \ldots \geq \lambda_{r,m}^*$ *the ordered solutions of the generalized eigenvalue problem*

$$\det\Big[V_{r,m}^* - \lambda(R_r^T D(1) D(1)^T R_r)^{-1}\Big] = 0. \tag{33}$$

Then the following convergence in distribution holds

$$n^2(\hat{\lambda}_{p-r+1,m}, \ldots, \hat{\lambda}_{p,m}) \to (\lambda_{1,m}^{*2}, \ldots, \lambda_{r,m}^{*2}).$$

Proof. Due to Lemmas 1, 2 and 4 (Bierens, 1997), it is sufficient to study the asymptotic behavior of $\sqrt{n} M_n^{nonst}$ and $n M_n^{st}$.
We have

$$\lim_{n \to +\infty} \sqrt{n} M_n^{nonst} = \lim_{n \to +\infty} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} F_k(t/n) \Delta^{d-1} Y_t +$$
$$+ \lim_{n \to +\infty} \int_{-\infty}^{d-1/2} \Big[\sqrt{n} \phi_1(n, \alpha) \sum_{t=1}^{n} G_{k,\alpha}(t/n) \Delta^{\alpha} Y_t\Big] \mathrm{d}\alpha =: L_1 + L_2$$

By Bierens (1997), we have to show that $L_2 = 0$.
Since $G_{k,\alpha} \in \mathcal{G}_{m,\alpha}$, then the existence of the function θ_1 (Definition 3.1-(i))

guarantees, that the Lebesgue Theorem on the dominate convergence holds. Therefore we can write

$$L_2 = \int_{-\infty}^{d-1/2} \lim_{n\to+\infty} \left[\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\right]\mathrm{d}\alpha.$$

Hence, the fractional lag-difference process $\Delta^{\alpha}Y_t$ is well defined. Definition **??**-(*ii*) assures that $L_2 = 0$, and the first part of the proof is complete.

Now,

$$\lim_{n\to+\infty} nM_n^{st} = \lim_{n\to+\infty} \sum_{t=1}^{n} F_k(t/n)\Delta^{d}Y_t +$$

$$+ \lim_{n\to+\infty} \int_{d-1/2}^{+\infty} \left[n\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\right]\mathrm{d}\alpha =: L_3 + L_4.$$

By Lemmas 1, 2 and 4 (Bierens, 1997), we need $L_4 = 0$.

Since $H_{k,\alpha} \in \mathcal{H}_{m,\alpha}$, the existence of the function θ_2 (Definition 3.1-(*ii*)) implies that the hypotheses of the Lebesgue's Theorem on the dominate convergence are fulfilled. Thus we have

$$L_4 = \int_{d-1/2}^{+\infty} \lim_{n\to+\infty} \left[n\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\right]\mathrm{d}\alpha.$$

The condition (*ii*) of the Definition 3.1 assures that $L_4 = 0$.

The result is completely proved. ■

4 THE GENERALIZED EIGENVALUE PROBLEM: THE DISCRETE CASE

The analysis carried out in the previous section deals with all differences of the fractional integrated process Y_t. This generality is implied by the continuous setting. However, to let this contribution be useful in computational economics applications, a discretization of the continuous case is provided. Such a discretization involves the M_n's described by (12) and (13), and it is made with respect to the difference order, named α, of the process Y_t. The discrete set of rational numbers $\mathbf{Q}$, that is infinite, countable and dense in $\mathbf{R}$ is used. The density property of $\mathbf{Q}$ in $\mathbf{R}$ permits to have a set of information not too restrictive, maintaining the model in line with the general features of the continuous case. Fix $k = 1, \dots, m$, where $m \in \mathbf{N}$. Let us consider F_k as in (7), and

$$\tilde{G}_{k,\alpha} : [0,1] \to \mathbf{R}, \qquad \alpha \in (-\infty, d - 1/2);$$

$$\tilde{H}_{k,\alpha} : [0,1] \to \mathbf{R}, \qquad \alpha \in (d - 1/2, d + 1/2).$$

Moreover, we define two sequences:

$$\{\zeta_1(n,\alpha)\} \subseteq \mathbf{R}, \qquad \alpha \in (-\infty, d - 1/2);$$

$$\{\zeta_2(n,\alpha)\} \subseteq \mathbf{R}, \qquad \alpha \in (d - 1/2, d + 1/2).$$

The terms M_n's defined in (12) and (13) can be rewritten as

$$M_n^{nonst} = \frac{1}{n} \sum_{t=1}^{n} F_k(t/n) \Delta^{d-1} Y_t + \sum_{j=1}^{+\infty} \Big[\zeta_1(n, \alpha_{1,j}) \sum_{t=1}^{n} \tilde{G}_{k,\alpha_{1,j}}(t/n) \Delta^{\alpha_{1,j}} Y_t \Big], \tag{34}$$

and

$$M_n^{st} = \frac{1}{n} \sum_{t=1}^{n} F_k(t/n) \Delta^{d} Y_t + \sum_{j=1}^{+\infty} \Big[\zeta_2(n, \alpha_{2,j}) \sum_{t=1}^{n} \tilde{H}_{k,\alpha_{2,j}}(t/n) \Delta^{\alpha_{2,j}} Y_t \Big], \tag{35}$$

where

$$\{\alpha_{1,j}\}_{j \in \mathbf{N}} \equiv \mathbf{Q} \cap (-\infty, d - 1/2)$$

and

$$\{\alpha_{2,j}\}_{j \in \mathbf{N}} \equiv \mathbf{Q} \cap (d - 1/2, d + 1/2).$$

Theorem 3.3 is translated in the discrete case.

Theorem 4.1 *Assume that $F_k \in \mathcal{F}_m$, $\tilde{G}_{k,\alpha_j} \in \mathcal{G}_{m,\alpha_j}$ and $\tilde{H}_{k,\alpha_j} \in \mathcal{H}_{m,\alpha_j}$ and Assumptions 2.1, 2.2 and 2.3 are true. Then the thesis of Theorem 3.3 holds.*

Proof. By the proof of Theorem 3.3, we have just to prove that

$$\tilde{L}_2 := \lim_{n \to +\infty} \sum_{j=1}^{+\infty} \Big[\sqrt{n} \zeta_1(n, \alpha_{1,j}) \sum_{t=1}^{n} \tilde{G}_{k,\alpha_{1,j}}(t/n) \Delta^{\alpha_{1,j}} Y_t \Big] = 0, \tag{36}$$

and

$$\tilde{L}_4 := \lim_{n \to +\infty} \sum_{j=1}^{+\infty} \Big[n \zeta_2(n, \alpha_{2,j}) \sum_{t=1}^{n} \tilde{H}_{k,\alpha_{2,j}}(t/n) \Delta^{\alpha_{2,j}} Y_t \Big] = 0. \tag{37}$$

By using Definition 3.1-(i), we have that

$$\tilde{L}_2 = \sum_{j=1}^{+\infty} \lim_{n\to+\infty} \Big[\sqrt{n}\zeta_1(n,\alpha_{1,j}) \sum_{t=1}^{n} \tilde{G}_{k,\alpha_{1,j}}(t/n)\Delta^{\alpha_{1,j}} Y_t\Big].$$

Therefore, by the fact that $\Delta^{\alpha_{1,j}} Y_t$ is independent on n, (ii) of Definition 3.1 implies that $L_2 = 0$.

Analogously, by using the conditions (iii) and (iv) in Definition 3.1, it is easy to show that (37) holds.

The proposition is completely proved. ■

The discrete case takes into account a countable, and infinite, number of differences of the process Y_t. Johansen (2005) proposes a fractional VAR model based on the d-th and 0-th differences of the data generating process. He analyzes the cofractionality of Y_t by using the reparametrization $\Delta^d Y_t$. In section 5 we present a VAR model to study the cofractional cointegration of $\Delta^d Y_t$ under some conditions on α. This model is based on a wider range of difference orders of Y_t than the Johansen's one.

5 A MODEL FOR NONPARAMETRIC FRACTIONAL COINTEGRATION ANALYSIS

In this section we propose a model for fractional integrated processes, in order to provide a natural field of application of the nonparametric cointegration theory developed in the previous sections.

Given $\alpha \in (-1/2, 1/2)$, we consider the $VAR_{d+\alpha,b_\alpha}(k)$ model in $(d+\alpha)$-th difference of the process Y_t:

$$\Delta^{d+\alpha} Y_t = (1-\Delta^{b_\alpha})\psi\xi^T \Delta^{d+\alpha-b_\alpha} Y_t + \sum_{i=1}^{k} \Gamma_i \Delta^{d+\alpha}(1-\Delta^{b_\alpha})^i Y_t + \epsilon_t, \tag{38}$$

where ϵ_t is i.i.d. in p-dimension with zero mean and finite variance Σ, ψ and ξ are $p\times r$ matrices, with $r < p$ and $b_\alpha > 0$ is the reduction in the order of integration. The concept of cointegration lies in the following definition.

Definition 5.1 *If $Y_t \sim I(d)$, with $d > -1/2$, and there exists a linear combination $\xi \neq 0$ such that $\xi^T Y_t \sim I(d-b)$, for $0 < b \leq |d|$, then Y_t is cofractional with cofraction vector ξ.*

ξ contains the r cointegrating vectors, quoted in Theorem 3.3. Therefore ξ is independent on the data generating process, and this fact justifies the nonparametric approach used for $VAR_{d+\alpha,b_\alpha}(k)$ in formula (40).

The model proposed in (40) is more general than the fractional VAR analyzed in the previous related literature. Johansen (2005) assume that $\alpha = 0$. In this way he works with the d-th difference of the process Y_t, that is an integrated process of order 0, independently on the value of d. In this particular case, the reduction in the order of integration b varies in a range independent on d. In our model α can be different from 0, and the reduction in the order of integration varies in a range dependent on α. Moreover, the definition of cofractional process in Definition 5.1 is more general than the one of Johansen (2005) and reported by several authors (see, as an example, Franchi (2007)). These authors assume that $0 < b \leq d$, working only on nonnegative integration orders $d \geq 0$. Our definition permits to consider processes with negative fractional order (restricted on (-1/2,0)), and falls in Johansen's definition for $d > 0$. Definition 5.1 allows for the following distinction for cofractional processes.

Assume that $\Delta^\alpha Y_t$ is cofractional with cofraction vector ξ. Then

- If $\Delta^\alpha Y_t \sim I(d-\alpha)$ is stationary, then also $\xi \Delta^\alpha Y_t$ is stationary of order less than $d-\alpha$.
- $\Delta^\alpha Y_t \sim I(d-\alpha)$ is nonstationary and $\xi \Delta^\alpha Y_t$ is stationary.
- $\Delta^\alpha Y_t \sim I(d-\alpha)$ is nonstationary and $\xi \Delta^\alpha Y_t$ is nonstationary of order in $(1/2, d-\alpha)$.

In the following result some conditions on the parameters α and b_α are provided, to summarize the previous distinction.

Proposition 5.1 Assume that $\Delta^\alpha Y_t$ is cofractional with cofraction vector ξ. If $\max\{0, d-\alpha-1/2\} < b_\alpha < d-\alpha+1/2$, then $\Delta^\alpha Y_t$ and $\xi^T \Delta^\alpha Y_t$ are stationary integrated processes.

Proof. Since $Y_t \sim I(d)$, then $\Delta^\alpha Y_t \sim I(d-\alpha)$ and, if $\Delta^\alpha Y_t$ is cofractional with cofraction vector ξ, then $\xi^T \Delta^\alpha Y_t \sim I(d-\alpha-b_\alpha)$. Under the stated hypotheses on α and b_α, we have

$$-1/2 < d - \alpha - b_\alpha < 1/2,$$

and the result is proved. ■

We provide some illustrative examples showing the sense of Proposition 5.1.

Example 5.1 We consider $Y_t \sim I(1/3)$. Let us work with the α-th difference of Y_t by choosing $\alpha = 1/6$. Then $\Delta^\alpha Y_t \sim I(1/6)$, and it is a stationary process. Then, for $b_\alpha = 1/6$, we have $\xi^T \Delta^\alpha Y_t \sim I(0)$.

Example 5.2 Assume $Y_t \sim I(8/9)$. For $\alpha = 1/9$, $\Delta^\alpha Y_t \sim I(7/9)$ and it is nonstationary.

- If $b_\alpha = 5/9$, we have $\xi^T \Delta^\alpha Y_t \sim I(2/9)$, and it is stationary.
- If $b_\alpha = 1/9$, we have $\xi^T \Delta^\alpha Y_t \sim I(2/3)$, and it is nonstationary.

The model proposed in (40) allows the estimate of the cointegration vectors of the process α-th difference of Y_t. Let us define $Z_t = \Delta^\alpha Y_t$. By introducing the random matrices A_m and B_m as in (8) and (9) for the process Z_t, then Theorem guarantees that such estimate is attained in a nonparametric framework.

6 CONCLUDING REMARKS

In this paper a nonparametric cointegration approach for fractional $I(d)$ process is proposed. In order to solve the generalized eigenvalues problem, two random matrices, taking into account the stationary and nonstationary part of the data generating process, are constructed. The solution of the problem is provided by assuming that the difference orders of Y_t vary in a continuous and discrete sets. The best feature of the continuous framework lies in its generality. The discrete case is proposed to let this contribution be useful in economics applications.

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