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A Discrete Model for Patent Valuation

by

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ABSTRACT

This article evaluates patents in a stochastic discrete time framework following the real options approach. By modeling the dynamics of the underlying as a spatial point process both size and time of the jumps can be treated as random variables. The propagation of the jumps from the underlying security to the patent value is not restricted to be immediate, but can occur with a random delay and with varying intensity, depending on the time to maturity. These actual features lead to a more generalized formula for patent value, that in turn may give rise to a non trivial difference in patent value, not accounted for in the existing literature.

Keywords: patent value; spatial mixed Poisson process; real options.

JEL Classification: O34, C65

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1 INTRODUCTION

This paper provides a theoretical contribution to the literature on patents valuation. Assigning economic value to patents is relevant for both patentees and at social level. Probably due to this twofold relevance, the economic valuation of patents has attracted much attention and efforts among economists insomuch that extensive documentation was provided since the second half of the '80s. In this paper we move from the common assumption that a patent is an option over a technology so that modeling can proceed as a derivative contract where the patent value is linked to the properties of the underlying security. Indeed, a patent gives the holder the exclusive right (option) on an invention. The right can be renewed to the expiration date T or be dropped before the statutory limit, if the holder decides not to pay the renewal fee. Yet, as patents have the three characteristics of: i) partial irreversibility of the investment undertaken, ii) market uncertainty, and iii) the possibility to delay the actions, a prominent part of the literature has regarded at the problem as one of the (real) options valuation (Pakes (1986), Pakes and Simpson (1989), Bloom and Van Reenen (2002), Laxam and Aggarwal (2003), Schwartz (2004), Baudry and Dumont (2006), Wu and Tseng (2006) among others).

The real options approach is based on the definition of an underlying security the evolution of which drives the patent evolution. The fluctuations in the underlying security capture the uncertainty affecting the supply and demand of goods, the production of which is entitled through patent ownership, research programmes from other firms, the attainment of patents, the changes in the regulatory environment, etc. Most of these factors governing the dynamics of the underlying security, do not occur continuously, but rather at discrete points in time, causing the underlying state to undergo a jump (for instance, it is quite unrealistic to assume that changes in the laws protecting patents occur continuously). This fact implies the presence of jumps in the dynamics of the patent value.

In spite of the relevance that jumps can take place in the real world, to the best of our knowledge, very little has been said in the literature. Some exceptions are worth mentioning: Schwartz (2004) models changes in regulatory environment as a possible jump in the Poisson process that the net cash flow generated by the project can undergo. In the spirit of McDonald and Siegel (1986) he considers only negative jumps affecting the value of the patent in two ways. First, reducing the expected rate of capital gain in the underlying security, which reduces the value of the option. Second, increasing the variance of percentage changes in the underlying security over finite time intervals, and this tends to increase the value of the option. The net effect is a reduction in the value of the option (patent). In a patent race context, jumps in a Poisson process are used to capture other firms patent attainment (Miltersen and Schwartz (2004), Lambrecht and Perraudin (2003), Weeds (2002)). According to a different approach, Baudry and Dumont (2006) give a definition of a very broad class of discrete time stochastic processes to describe the evolution of the rent generated by a patent. In such a way they model regulatory interventions, such as a change in the schedule of renewal fees, highlighting how the profile of renewal fees can reduce applications from worthless patents.

Within the real options approach we will present a very general model meant to provide a unifying theoretical framework where the existing models of patent valuation can be considered as special cases. The generality of the model also allows for unconsidered theoretical hypotheses, taking a step forward towards a more realistic model. As we will see, this can be achieved by incurring the cost of adopting an unusual process: the Spatial Mixed Poisson Process, SMPP.

Is it worth incurring this cost? The answer must be given both in terms of theoretical advances the model can generate and in terms of economic implications.

As to the theoretical aspects, the approach can produce several new results according to the existing literature. First of all, it allows tackling the problem of the value of patents in the presence of jumps in the underlying process allowing the process to undergo both negative and positive jumps. The latter can accommodate more effective or rigorous protection policies, such as the establishment of the Court of Appeals of the Federal Circuit by US Congress or the EU directive 2004/48 on the enforcement of intellectual property rights. Secondly, the renewal process is not contemplated in the above-mentioned literature, which models (only negative) jumps through a Poisson process. At the same time, the renewal process has been extensively modeled in literature (from Pakes (1986) onwards), but dropouts have been modeled under the "no jumps" assumption. Our general model puts together the two strands of literature, modeling the jumping value of a patent in the presence of a renewal schedule. Thirdly, thinking of a negative jump due to other firms attainment, it can happen, and indeed it does, that the novelty contained in the others' patent is not so crucial, being, let us say, an invention around attainment. In this case, the patent under-valuation suffers from a negative jump but the jump may not be large enough to make the patent worthless. Nevertheless, not considering it would be inadequate. Differently from any other paper in the field, the entity of the jump is not known in advance according to our approach. As a consequence, and even more importantly, the killing jump is not restricted to the first negative jump that may occur, but only a large enough jump can kill the patent. Empirical works (Schankerman (1998), Lanjouw et al. (1998)) report that the distribution of the value of patents is highly skewed, as well as that there are many patents with low value and just a few with a high one. It has been sensibly assumed that when a low-valued patent is taken out by a competing firm, the high-valued patents undergo a non deadly jump. Therefore, according to the quoted papers, the number of these jumps is quite far from being negligible and, consequently, the cumulated change in the high-valued patent is not marginal too. Fourthly, there is a random delay in the transmission of the jump from the underlying state to the patent value. As described hereafter, this plays an important role in modeling patents with different jump behaviours. Again, let us consider a negative jump due to other firms' patent attainment. It has never been clarified in the existing literature why other firms' attainment should bring about an immediate obsolescence in the patent that is being valued. To fix ideas, think of a new patent in the consumer electronics field: it takes time before the new product is marketed and, once marketed, it takes time before it is sufficiently widespread to cut to nil the profit accruing from the "old" patent. Yet, once the product is marketed, there is a non negligible part of consumers, typically elderly people, who show a high degree of stickiness in accepting new technologies, making still temporarily worth the old ones. In such a case, the assumption of a no-delay condition is a naïve approximation. At the same time, there can be also circumstances in which the delay is almost nil. For instance, it may be that clinical trials reveal that a patented drug has some terrible side effects, or it may be that the government prohibits certain classes of drugs. This case too is accounted for by the model by restricting the transmission delay of the jump to zero, as a special case. Fifthly, as the patent gets older, namely approaching the final date T, its value becomes less sensitive to the jumps in the underlying security. That is because the propagation can take place progressively rather than abruptly and also because some jumps in the underlying security do not have enough time to propagate to the patent value. Therefore, we can realistically affirm that the sensitivity of the patent value decreases with respect to the time of the jumps. As a by-product, this sort of decay in the strength of the jumps – associated to the delay in the transmission mechanism - allows to capture the limiting cases in which the jump in the underlying security is not transmitted to the patent because too close to expiry. Sixtly, the adoption of the SMPP avoids the unpleasant hypothesis of no expiration date (Schwartz and Moon (2000), Bloom and Van Reenen (2002)), thus providing the value of the patent in a closed form solution, and not as a numerical solution to a partial differential equation.

The present article investigates the patent value from the private point of view, Nevertheless, the economic implications arising from the paper show that a number of policy implications are attached to the analysis. The delay in the transmission mechanism plays a role when determining a selling or licensing price. If the parties do not consider it properly, they take the risk of over (under) estimating the patent if negative (positive) jumps in the underlying state have already occurred without propagating on the patent value yet. The same fault might be repeated when estimating the aggregate value of patent rights and when using such estimation to assess the importance of patent protection. Patent protection, related to other systems of return-appropriation from inventions, is a crucial issue for policy makers to spur innovation and, more generally, R&D. Nevertheless, if the value of patent rights is inadequately estimated, biased incentives will be fostered. Yet, as shown hereafter, comparative studies on patent rights valuation are not necessarily affected by the same bias, making the comparison meaningless. The entity and sign of the distortion are hard to predict. Little can be said a priori. Caeteris paribus, the transmission delay will depend on the technology field, the average population age, the firms' responsiveness to changes and on many other unpredictable factors. For instance, a country with a relatively large number of patents in the pharmaceutical field will have a shorter average delay at aggregate level, with respect to another country strongly specialized in consumer electronics. At the same time, for given technological specialization, "younger" countries are expected to experience shorter delays. Moreover, when the occurrence of positive jumps is not accounted for by the model and when the first jump is considered to be deadly, the resulting patent value is definitely underestimated, other things equal. Similar reasoning apply to at micro level.

Quantifying the potential bias in the patent valuation goes far beyond the scope of this paper. Nonetheless, the main message from the model is that, in principle, the valuation can drastically differ and policy recommendations can be mistakenly inferred if one neglects the possibility of positive jumps, and the delay and decay in jumps propagation. Neither a sensible comparison of patent rights can be made between countries and sectors.

It is important to highlight that the cost to incur in terms of algebra to enjoy the benefits is not sunk, in the sense that the Mixed Poisson Processes on the line and the infinite-server queue models are widely known and used (see e.g. Grandell (1997)); and also a spatial setting can be useful in modeling several practical situations (e.g. spatial queues, see e.g. Cinlar (1995)).

From a purely technical point of view, we focus on SMPPs to develop our theoretical model by using stochastic techniques grounded on some remarkable invariance properties of the family of such processes. The dynamics of the underlying state is assumed to evolve in discrete time and both sizes and times of the jumps are assumed to be stochastic variables. A discrete environment is in agreement with Baudry and Dumont's (2006) model, but we depart from their binomial tree by moving to an approach involving Mixed Poisson Processes.

In our setting, a spatial point process is given $R := \{(\tau_i, \xi_i)\}_{i \in \mathcal{N}}$, by where the coordinates τ_i and ξ_i represent, respectively, the arrival time and the size of the *i-th* jump in the dynamics of the underlying security. We also assume that the random variables ξ_i are i.i.d. and independent from the $\{\tau_i\}_{i \in \mathcal{N}}$ one-dimensional process . More specifically, at time τ_i a jump of size ξ_i occurs to the dynamics of the underlying security. After a delay, the value of the patent jumps as well, with an intensity dependent on ξ_i and τ_i. The delay is considered to be random in that, as already put forward, it depends on several factors, not completely under the economic actors' control. Denoting by N the transformed point process, namely the value of the derivative, the type of correlation between R and N imposes to treat R as a Spatial Point Process. This distributional hypothesis on R guarantees theoretical results in agreement with the economic intuition. Let us elaborate this issue. To be more consistent from an economic point of view, we assume the lack of independence between delays and jump sizes. This reasonable hypothesis imposes to abandon the usual Mixed Poisson Process framework and to treat jump arrivals, τ_i, as Spatial Processes. Indeed, if jump arrival times follow a one-dimensional Mixed Poisson Process, jump sizes, ξ_i are i.i.d. and independent on the arrival times. Moreover, in such a case the delays are independent on R. It follows that R and N have the same stochastic structure. As argued in the paper, when the delays are not independent on R, the invariance property does not hold, but it can be re-obtained by assuming that jump arrival times follow a SMPP. In particular, some recent results on stochastic structure invariance of SMPPs with respect to a class of random transformations (see Foschi and Spizzichino (2008)) will turn to be very useful in deriving the final results.

This set up allows to obtain a theoretical estimate of patent value by providing a valuation of the total size of its jumps in a given period. At this point, a clarification of the meaning of "theoretical estimate" is in order. Since a patent jumps with a certain delay, the jumps that will occur tomorrow are partially due to the jumps in the underlying security that have occurred today. Hence, to value the patent tomorrow one must give and "estimate" of the jumps in the underlying security already occurred, the effect of which will be exerted tomorrow. In this sense, and not in the econometric one, the expression "theoretical estimate" must be interpreted throughout the paper.

This quantity is worked out taking also into account the economic and statutory constraints that patents are subject to. It can be considered as a key quantity to compare our general result with those obtained under more restrictive hypotheses.

The remaining part of the paper is organized as follows. Section 2 presents the set up of the model describing the dynamics of the underlying security, constrained by some economic and statutory terms which patents are subject to. Section 3 presents the theoretical results on the valuation of a patent over a given time period. Section 4 endogenizes the renewal threshold and presents the valuation mechanism for valuing a patent at a given point in time. Finally, Section 5 summarizes the results and provides the conclusion. The main definitions and the key results on SMPPs are contained in the Appendix.

2 THE MODEL

The model for valuing a patent presented hereafter is in accordance with the real options approach. The value of a patent is supposed to be driven by the evolution of an underlying state. We stress that the evolution of patent value can drastically change in the presence of impulsive events. A technological improvement or the introduction of laws modifying the protection rules can be reasons for a jump in the dynamics involved in the patent valuation process. Therefore, in this model we rely on an underlying process driving the patent value in a discrete time framework. To this respect, it is interesting to note that some models of patent valuation approximate the results obtained in continuous time with an approximate formulation in discrete time, recognizing the latter to be more realistic. *"[...]* The decision to abandon the project is evaluated at discrete points in time, instead of continuously. This would seem to be a more reasonable assumption when analyzing R&D projects [...]" (Schwartz (2004) p. 52)

Let us consider a probability space with filtration $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\geq 0}, P)$ containing all the random variables used throughout the paper. Let us denote as T the set of the stopping times in $[0,+\infty]$, i.e.

$$\mathbf{T} := \{\tau : \Omega \to [0,+\infty] | \{\tau \leq t\} \in \mathbf{F}_t, \ \forall t \geq 0\}. \tag{1}$$

The time-dependent dynamic S_t of the underlying security related to a patent is assumed to jump as follows:

$$S_t = S_0 + \sum_{i=1}^{+\infty} \mathbf{1}_{\{\tau_i < t\}} \xi_i, \tag{2}$$

where: $S_0 > 0$ is the initial value of the underlying security. It is reasonably known, so we assume that S_0 is a deterministic positive constant.

$\tau_i \in \mathbf{T}$, $i \in \mathcal{N}$, and τ_i takes on values in $[0,+\infty]$. It is the stochastic time of the i -th jump in the dynamics of the underlying security.

$\xi_i \in \mathbf{F}_{\tau_i}$, $i \in \mathcal{N}$, and ξ_i takes on values in $\mathcal{R}$. It represents the size of the i-th jump of the underlying state dynamics.

The patent valuation theory assumes that the value of a patent is null when the underlying state value is equal to zero. In turn, the underlying state is null when a negative jump with a *large enough* size occurs. From these simple considerations the following result states immediately:

Proposition 1

Define τ^* *as the first hitting time of* S_t *on the boundary 0 :*

$$\tau^* := \inf\{t \geq 0 | S_t = 0\}. \tag{3}$$

Then one of the following alternatives holds:

- $\tau^* = +\infty$
- $\exists j \in \mathcal{N} \,|\, \tau^* \equiv \tau_j$

Moreover, patents are subject to a statutory limit, i.e. the *expiration date* $T > 0$, when the patent loses its value and the production of the protected good becomes of public domain[1]. Hence, the underlying state must be observed till

[1] Generally, T=20 years.

the threshold T if the dynamics do not hit the absorbing barrier zero before. This information can be synthetically formalized as

$$S_t = \begin{cases} S_0 + \sum_{i=1}^{+\infty} \mathbf{1}_{\{\tau_i < t\}} \xi_i, & \text{for } t < \tau^* \wedge T; \\ 0, & \text{for } t \geq \tau^* \wedge T. \end{cases} \tag{4}$$

The underlying state dynamics can be fully described by using a point process framework. The set of bivariate random variables $R := \{(\tau_i, \xi_i)\}_{i \in \mathrm{N}}$ is a spatial point process for S_t, where the random variables ξ_i are i.i.d. and independent from the one-dimensional process $\{\tau_i\}_{i \in \mathrm{N}}$.

In our setting, we will consider $\xi_i > 0$ as a *good news* for the patent value, and $\xi_i < 0$ as a *bad news*. For instance, *good news* can be represented by any policy aimed at strengthening or widening the patent holder's rights, such as a more efficient judicial system, a decrease in costs of suing an infringer, an enhancement of the enforcement system, a tougher infringer's punishment, an increase in the statutory limit, etc. Shortly, *good news* are such to produce additional opportunities to exploit the patented innovation, conversely for *bad* news (Baudry and Dumont (2006)).

We now stress that the presence of jumps in the dynamic of the underlying security implies the presence of jumps in the value of the patent. We describe this mechanism by introducing a bivariate random variable $w_i = (w_i^{(1)}, w_i^{(2)})$, $i \in \mathcal{N}$, taking values on a set $\mathbf{W} \subseteq \mathcal{R}^2$, and a transformation ϕ,

$$\phi : [0,+\infty] \times \mathcal{R} \times \mathbf{W} \rightarrow [0,+\infty] \times \mathcal{R},$$

such that $\phi(\cdot,\cdot,w_i) : [0,+\infty] \times \mathcal{R} \rightarrow [0,+\infty] \times \mathcal{R}$ is measurable and one-to-one for any fixed $w_i \in \mathbf{W}$. Now, we will consider the point-wise spatial transformation Φ_ϕ of the point process R and the variable $\mathbf{W}$:

$$N \equiv \Phi_\phi(R, \mathbf{W}) = \{\phi(\tau_i, \xi_i, w_i)\}_{i \in \mathcal{N}}, \tag{5}$$

where $\mathbf{W} = \{w_i\}_{i \in \mathcal{N}}$. The random quantity $\Phi_\phi(R, \mathbf{W})$ models the jumps of the value of the patent accruing from the jumps in the underlying security dynamics. Following the aforementioned literature on renewals, we must also insert in the dynamics of patent value the fee that a patent holder periodically pays to keep alive the patent itself. Indeed, a patent holder pays a known positive amount

X_j, in general not constant, at a fixed date T_j, with $j \in \mathcal{N}$. This deterministic payment process stops naturally in three situations:

- when the expiration date T is reached;
- after the random time τ^*, e.g. when the value of the underlying security becomes null. In this case, obviously, the last periodical payment is the one immediately before the exit time τ^*;
- if the patent holder reckons that patent renewal is not a suitable economic strategy. This can happen if the expected net gain from holding the patent is negative. Formally, for the time being, we can assume that there exists a deterministic time-varying threshold $\gamma(t)$ such that $X_k = 0$, for each integer $k > j^*$, where

$$j^* := \min\{j \in \mathcal{N} \mid X_j \geq \gamma(T_j)\}. \tag{6}$$

We will refer to $\gamma(t)$ as the *renewal threshold* and it will be endogenized in section 4. In order to formalize the intervention of the critical index j^* in our model, we define the time-dependent *property* Π_t as $\Pi_t := \{T_{j^*} \leq t\}$.

Indicating the renewal time at which the patent holder decides not to renew the patent. When the patent is not renewed, its value obviously becomes null from that time onward. In particular, when the underlying security reaches the expiration date *T* or the patent is not renewed, the value of the patent becomes immediately null, without time transformations. Differently, when the underlying security dynamics reaches the barrier 0, the patent value becomes null at the random time-transformation of τ*. Furthermore, we denote the dynamics of patent value in the time interval $[0,t]$ as $C([0,t])$. We also denote as C_0 the starting point of patent dynamics at time $t = 0$. As it should be, C_0 is a deterministic nonnegative term. Lastly, we denote as $\phi_{|\tau_i}$ the restriction of the function ϕ to the first component. In this set up several cases can be distinguished.

- If $t < \tau^* \wedge T$ and property Π_t does not hold, then

$$C([0,t]) = C_0 + \sum_{i=1}^{+\infty} \phi(\tau_i, \xi_i, w_i) \mathbf{1}_{\{\phi_{|\tau_i}(\tau_i)\ \le\ t\}} - \sum_{j=1}^{+\infty} X_j \mathbf{1}_{\{T_j\ \le\ t\}}; \tag{7}$$

  the value of a patent over the interval $[0,t]$, when a *large enough* jump in the underlying security has not occurred, $t < \tau^*$, and the final expiration date, T, has not been reached is given by the algebraic sum of three components: the initial value of the patent, C_0, plus the sum of the changes in the patent value accruing from the jumps in the underlying security, $\phi(\cdot,\cdot,\cdot)$, minus the sum of all the renewal fees paid till time t.

- if $\tau^* \le t < T$ and property Π_{τ^*} does not hold, then

$$C([0,t]) = C_0 + \sum_{i=1}^{+\infty} \phi(\tau_i, \xi_i, w_i)\ \mathbf{1}_{\{\{\phi_{|\tau_i}(\tau_i)\ \le\ t\}\cap\{\tau_i \le \tau^*\}\}} - \sum_{j=1}^{+\infty} X_j \mathbf{1}_{\{T_j\ \le\ \tau^*\}}; \tag{8}$$

  equation (8), consistently with (7), redefines the value of a patent in the occurrence that a *large enough* jump in the underlying security had intervened before the final date and the patent holder has paid the renewal fees till τ^*

- if $T \le t < \tau^*$ and property Π_T does not hold, then

$$C([0,t]) = C_T \ := C_0 + \sum_{i=1}^{+\infty} \phi(\tau_i, \xi_i, w_i)\ \mathbf{1}_{\{\phi_{|\tau_i}(\tau_i)\ \le\ T\}} - \sum_{j=1}^{+\infty} X_j \mathbf{1}_{\{T_j\ \le\ T\}}; \tag{9}$$

  equation (9) depicts the situation in which a patent reaches the final expiration date, T, and its value is determined by the whole set of jumps in the underlying security. Therefore, we avoid the unpleasant hypothesis of no expiration date (as in Schwartz and Moon (2000), Bloom and Van Reenen (2002)).

- if property Π_t holds (and obviously $t \le \tau^*$), then

$$C([0,t]) = C_{T_{j^*}} \ := C_0 + \sum_{i=1}^{+\infty} \phi(\tau_i, \xi_i, w_i)\ \mathbf{1}_{\{\phi_{|\tau_i}(\tau_i)\ \le\ T_{j^*}\}} - \sum_{j=1}^{j^*} X_j . \tag{10}$$

  eventually, (10) depicts the early exit at a renewal date decided by the patent holder not paying the renewal fee.

An explicit formulation of the transformation ϕ can be finally introduced and substituted into equations (7-10) describing the patent value. To this aim, we highlight a few important aspects to make the valuation process as generalized as possible:

- According to the real options literature on patent valuation, the presence of a jump in the underlying security is meant to capture some stylized facts, for instance, when another firm takes out a patent on a new, more technologically-sophisticated product. The new product is supposed to make the valued patent immediately outdated. In the real world it takes time before the new product is marketed and, once marketed, it takes time before it is widely diffused to the point of cutting to zero the profits accruing from the old patent. According to the model, it takes time before a jump in the underlying security turns into a corresponding jump in the patent value. For instance, let's think of a new patent for some electronic device, such as a new cellphone or a new television incorporating technology improvements. It is quite evident that it takes time between the new patent registration and the complete obsolescence of the old patent. Especially for consumer electronics, elderly people are less willing to adopt new technologies, making still temporarily worth the old ones. In such a case, the assumption of a no-delay condition is a naïve approximation. At the same time, we do not deny that there can be circumstances in which the delay is almost null. For instance, it may be that clinical trials reveal that a patented drug has some terrible side effects, or it may be that the government prohibits certain classes of drugs (Miltersen and Schwartz (2004)). In order to make the transmission mechanism of the jump from the underlying security to the patent value as general as possible, we assume that there is a random delay between the time of occurrence of a shock in the underlying state dynamics and the time of the correspondent jump in the patent value. The no delay case can thus be regarded as a special case, embedded into this very general assumption. This general transmission mechanism has practical spillovers when determining licensing or selling prices. If the parties do not take the transmission delay into account, the deal can finally set a too high (low) price whenever a negative (positive) jump in the underlying dynamics occurs and is not yet propagated to the patent. At aggregate level, macro valuations of patents rights can end up with over/under estimates. It follows that if important patent-protection measures, concerning other means of returns-appropriation from invention, are constructed[2], misleading policy

[2] Such as the Equivalent Subsidy Rate, ESR, given by the ratio of the total value of patent rights relative to R&D used to produce those patents.

implications can be drawn and misleading comparisons can be made between countries and sectors.

- The value of a patent falls (rises) when a bad (good) news appears in the economic system, i.e. the presence of a jump of negative (positive) size. The entity of the rise or fall of the patent value is not fully controlled by the economic actors, i.e., the patentees.
- The value of the patent is increasing with respect to the size of the jumps in the underlying security.
- The propagation process can be progressive rather than abrupt and some jumps in the underlying security have not enough time to propagate to the patent value. Therefore, when modeling the realistic feature, we also consider that the sensitivity of the patent value decreases with respect to the time of the jumps. Formally, if two jumps of the same size occur at two different points in time, say $\tau^* < \tau^{**}$ the value of the patent has a more remarkable jump in τ^*, since the end of the patent's life is nearer. As a by-product, this sort of decay in the strength of the jumps, associated to the delay in the transmission mechanism, allows to capture those limiting cases in which the jump in the underlying security is not transmitted to the patent because too close to expiry. For instance, suppose a rival firm takes out a new patent when the old one is close to expiration. The expected profit loss will be almost negligible, both because of the residual short life of the patent and because of the delay in the jump propagation to the patent value.

Considering the above**,** we assume that ϕ can consistently operate as follows:

$$\phi(\tau_i, \xi_i, w_i) := (\tau_i + w_i^{(1)}, w_i^{(2)} \xi_i \mathrm{e}^{-\tau_i}), \quad i \in \mathcal{N}, \tag{11}$$

where $w_i^{(1)}, w_i^{(2)}$ are random variables with nonnegative support, i.e. $\mathbf{W} \equiv [0, +\infty)^2$ representing a stochastic delay and a stochastic percentage, respectively. They translate times and sizes of the jumps in the dynamics of the underlying process into times and sizes of the jumps in the patent value. By defining

$$\lambda_i := \tau_i + w_i^{(1)}, \quad \gamma_i := w_i^{(2)} \xi_i \mathrm{e}^{-\tau_i}, \quad \forall\, i \in \mathcal{N} \tag{12}$$

the transformed process N defined in (N) can be rewritten as

$$N = \{(\lambda_i, \gamma_i)\}_{i \in \mathrm{N}}. \tag{13}$$

Thus, the evolution of patent value in (7-10) can be rewritten as:

If $t < \tau^* \wedge T$ and property Π_t does not hold, then

$$C([0,t]) = C_0 + \sum_{i=1}^{+\infty} \gamma_i \mathbf{1}_{\{\lambda_i \le t\}} - \sum_{j=1}^{+\infty} X_j \mathbf{1}_{\{T_j \le t\}}. \qquad (14)$$

if $\tau^* \le t < T$ and property Π_{τ^*} does not hold, then

$$C([0,t]) = C_0 + \sum_{i=1}^{+\infty} \gamma_i \mathbf{1}_{\{\lambda_i \le t\} \cap \{\tau_i \le \tau^*\}\}} - \sum_{j=1}^{+\infty} X_j \mathbf{1}_{\{T_j \le \tau^*\}}; \qquad (15)$$

if $T \le t < \tau^*$ and property Π_T does not hold, then

$$C([0,t]) = C_T := C_0 + \sum_{i=1}^{+\infty} \gamma_i \mathbf{1}_{\{\lambda_i \le T\}} - \sum_{j=1}^{+\infty} X_j \mathbf{1}_{\{T_j \le T\}}; \qquad (16)$$

- if property Π_t holds (and, obviously, $t \le \tau^*$), then

$$C([0,t]) = C_{T_{j^*}} := C_0 + \sum_{i=1}^{+\infty} \gamma_i \mathbf{1}_{\{\lambda_i \le T_{j^*}\}} - \sum_{j=1}^{j^*} X_j. \qquad (17)$$

To the aim of providing an estimate of the patent value (14-17) we can assert:

Remark 1 *Even if the sizes of the stochastic jumps are theoretically unbounded, without loss of generality one can reasonably assume the existence of a positive upper and a negative lower threshold sufficiently large for the jumps sizes in the underlying state dynamics and patent value. Therefore, we assume hereafter the existence of two positive constants* a *and* b *such that, fixed* $j \in \mathcal{N}$ *,* ξ_i *and* γ_i *are random variables with support in* $[-a,a]$ *and* $[-b,b]$*, respectively.*

There is a clear correlation between γ_i and the (τ_i, ξ_i) couple. As already argued in the Introduction, this fact does not allow to consider *R* as a stochastic process on the line, and we need to treat *R* as a spatial point process. The assumption that *R* is a SMPP allows to reach two targets. First of all, to provide a model for simultaneously estimating the number and size of the jumps in the dynamics of the underlying security and, consequently, the number and size of the jumps in the dynamics of patent value. Secondly, to show that, according to some recent results, SMPPs can guarantee the invariance of the stochastic structure between *R* and *N*. More precisely, if *R* is a SMPP, the *N* process is a SMPP too (see the Appendix).

3 THE PATENT VALUE OVER A FINITE TIME INTERVAL

This section is devoted to provide a mechanism for the valuation of a patent in a given period. The valuation is attained by computing the total size of the jumps in the underlying security over a given time interval

$$\mathbf{I} := [T', T' + s]$$

with $T', s > 0$ and $T' + s \leq T$, and then providing an estimate of the corresponding total size of patent jumps that will occur in the following time interval

$$\mathbf{H} = [T'', T'' + r],$$

with $r > 0$, $T'' = T' + s$. Without loss of generality, we can assume that $T'' + r \leq T$. $\mathbf{I}$ is the period *under scrutiny* and $\mathbf{H}$ is the time period that we observe, assumed to be sequential to $\mathbf{I}$. We then consider the jumps occurring at times $\tau_i \in \mathbf{I}$. In particular, if a shock intervenes in the economic environment and the underlying security dynamics register a jump of size ξ_i at a random time $\tau_i \in \mathbf{I}$, then the same shock propagates on the value of the patent, as explained through the introduction of function ϕ. More precisely, after the delay λ_i, the patent value registers a jump of size. γ_i. We now consider the regions $I \equiv \mathbf{I} \times [-a, a]$, $J \equiv \mathbf{I} \times [-b, b]$ and $H \equiv \mathbf{H} \times [-b, b]$. In agreement with a commonly used notation, we denote by $R(\Delta)$ and $N(\Delta)$ the number in the elements of the spatial point process R and N respectively, that are contained in a bidimensional set Δ. Further, let us define the random subset of indexes $\{i_1, ..., i_K\} \subset \mathcal{N}$, such that

$$\{(\tau_{i_1}, \xi_{i_1}), ..., (\tau_{i_K}, \xi_{i_K})\} = I \cap R.$$

We notice that $K \equiv R(I)$.

The exit time of the underlying security dynamics from the barrier 0 is located at a random jumping time $\tau_{i_{\overline{K}}}$, by Proposition 1, where $\overline{K} \in \mathcal{N}$, with the convention $\overline{K} := +\infty$ if S_t does not hit the barrier 0. We denote by I_{τ^*} the restriction of the counting spatial process R in the set I up to the stochastic threshold τ^*. Formally, we have

$$\{(\tau_{i_1}, \xi_{i_1}), ..., (\tau_{i_{n \wedge \overline{K}}}, \xi_{i_{n \wedge \overline{K}}})\}_{n=1...,K} = I_{\tau^*} \cap R, \text{ and } K \wedge \overline{K} \equiv R(I_{\tau^*}).$$

Equally, let us denote the random subset of indexes $\{i_1,...,i_{\tilde{K}}\} \subset \mathbb{N}$, such that $\{(\lambda_{i_1},\gamma_{i_1}),...,(\lambda_{i_{\tilde{K}}},\gamma_{i_{\tilde{K}}})\} = J_{\tau^*} \cap N,$ where J_{τ^*} denotes the propagation on the process N in the set J of the restriction of R due to the presence of the exit time τ^*. We notice that $\tilde{K} \equiv N(J_{\tau^*})$.

Furthermore, property Π_t implies the existence of an index $\tilde{K}^* \le \tilde{K}$ such that $\tilde{K}^* < j^* \le \tilde{K}^* + 1$. We denote as $J_{\tau^*,*}$ the further restriction on J_{τ^*} driven by property Π_t. We have $\{(\lambda_{i_1},\gamma_{i_1}),...,(\lambda_{i_{\tilde{K}^*}},\gamma_{i_{\tilde{K}^*}})\} = J_{\tau^*,*} \cap N.$

We also consider the random subset of indexes $\{i_1',...,i_{K'}'\} \subseteq \{i_1,...,i_{n\wedge \overline{K}}\}_{n=1,...,K}$ such that the delay λ_i, with $i \in \{i_1',...,i_{K'}'\}$, falls in the set J. We have a conditioning on the set I_{τ^*}. Taking into account property Π_t, we finally define:

$$\{(\lambda_{i_1'},\gamma_{i_1'}),...,(\lambda_{i_{\tilde{K}}'},\gamma_{i_{\tilde{K}}'})\} \cap \{(\lambda_{i_1},\gamma_{i_1}),...,(\lambda_{i_{\tilde{K}^*}},\gamma_{i_{\tilde{K}^*}})\} = J_{\tau^*,*} \cap N \,\Big|\, I_{\tau^*}.$$

We denote $K' \wedge \tilde{K}^* \equiv N_{(I_{\tau^*})}(J_{\tau^*,*})$ .Evidently, $P(N_{(I_{\tau^*})}(J_{\tau^*,*}) \le R(I_{\tau^*})) = 1.$

Remark 2 *Since a SMPP is a simple process (see Appendix, Lemma 1) and regions* I_{τ^*} *and* $J_{\tau^*,*}$ *are bounded, then* $\mathbb{E}[R(I_{\tau^*})] < +\infty$ *and* $\mathbb{E}[N(J_{\tau^*,*})] < +\infty$.

We now provide an estimate of the total size of patents jumps due to the jumps in the underlying security in $\mathbb{H}$, following the Cerqueti et al. approach (2009). Let $\Upsilon = \{j_1,..,j_{K''}\}$ be the random subset of indexes such that $\{(\lambda_{j_1},\gamma_{j_1}),...,(\lambda_{j_{K''}},\gamma_{j_{K''}})\} = H_{\tau^*,*} \cap N.$ As usual, $H_{\tau^*,*}$ denotes the restrictions on the process N in H due to the presence of the threshold τ^* for R and the property Π_t for N. By definition of the process N, we can also write $K'' \equiv N(H_{\tau^*,*})$.

We omit hereafter the subscript τ^* in I and $\tau^*,*$ in J and H to have a less cumbersome notation. Nevertheless, the computed quantities have to be intended *under the threshold condition* τ^* *for the process* R *and under the* *property* Π_t *for the process* N. The amount $Q_{\mathbf{H}}$ represents the total size of the jumps in the patent value over the time interval $\mathbf{H}$, due to the jumps in the underlying security and it is defined as $Q_{\mathbf{H}} \equiv \sum_{j \in \Upsilon} \gamma_j$.

A theoretical valuation of the entity $Q_{\mathbf{H}}$ can be obtained on the basis of the information collected in the previous period $\mathbf{I}$. More precisely, we will approximate the conditional expectation of $Q_{\mathbf{H}}$ given:

- the number $R(I)$ of jumps in the underlying security dynamics during the interval $\mathbf{I}$;
- the number $N(J)$ of jumps in the patent value during $\mathbf{I}$;
- the number of jumps in the underlying security occurred over $\mathbf{I}$ that have propagated to the patent in the same time interval. According to formula (33), we denote this quantity by $N_{(I)}(J)$.

We consider a partition of H :

$$\Delta_k \coloneqq \{H_s^{(k)}\}_{s=1,\dots,k}, \quad k \in \mathcal{N}, \tag{18}$$

where $H_s^{(k)} \coloneqq \mathbf{H} \times (c_{s-1}^{(k)}, c_s^{(k)}]$, with $s = 1,\dots,k$, $c_0^{(k)} = -b,\ c_k^{(k)} = b$ and, for each k, $\{c_s^{(k)}\}$ is increasing with respect to s.

We denote by $a_s^{(k)}$ the expected number of jumps in patent value observed in the time interval $\mathbf{H}$ and with size $(c_{s-1}^{(k)}, c_s^{(k)}]$, for each $s = 1,\dots,k$, conditioned on the previous history in the period $\mathbf{I}$, i.e.

$$a_s^{(k)} \equiv \mathsf{E}\Big[N(H_s^{(k)}) \mid R(I) = n', N(J) = n'', N_{(I)}(J) = m \Big]. \tag{19}$$

The next result provides a closed form expression to compute $a_s^{(k)}$, for any $k \in \mathcal{N}$ and $s = 1,\dots,k$. Evidence is given in Cerqueti et al. (2009).

Proposition 2

$$a_s^{(k)} = \sum_{n=0}^{+\infty} \sum_{l=0}^{n} n \frac{[M_{(\bar{I})}^*(H_s^{(k)})]^{n-l}}{(n-l)!} \frac{[M_{(I)}^*(H_s^{(k)})]^{l}}{l!} \int_0^{+\infty} \lambda^{n-l} e^{-\lambda M_{(\bar{I})}^*(H_s^{(k)})} u(\lambda; I, J, n', n'', m) \mathbf{d}\lambda \cdot$$

$$\cdot \int_0^{+\infty} \lambda^{l} e^{-\lambda M_{(I)}^*(H_s^{(k)})} u(\lambda; I, J, n', n'', m) \mathbf{d}\lambda,$$

where

$$u(\lambda; I, J, n', n'', m) = \frac{\lambda^{n''-m+n'} e^{-\lambda[M^*_{(\bar{I})}(J)+M(I)]} u(\lambda)}{\int_0^\infty \lambda^{n''-m+n'} e^{-\lambda[M^*_{(\bar{I})}(J)+M(I)]} u(\lambda) d\lambda}$$

is the posterior distribution on Λ defined in Eq. (35).

By using Proposition 2, we can provide an upper and a lower approximation of $\mathsf{E}[Q_{\mathbf{H}} \mid R(I) = n', N(J) = n'', N_{(I)}(J) = m]$. In fact, by letting

$\overline{\theta}_k := \sum_{s=1}^{k} c_s^{(k)} a_s^{(k)}$, $\hat{\theta}_k := \sum_{s=1}^{k} c_{s-1}^{(k)} a_s^{(k)}$ and for any $k \in \mathcal{N}$, we have

$$\hat{\theta}_k \leq \mathsf{E}[Q_{\mathbf{H}} \mid R(I) = n', N(J) = n'', N_{(I)}(J) = m] \leq \overline{\theta}_k. \tag{20}$$

$\{\hat{\theta}_k\}$ is non-decreasing and $\{\overline{\theta}_k\}$ is non-increasing with respect to k. Moreover, there exists a nonnegative constant depending on the region $\mathbf{H}$, say q_{H}, such that $\lim_{k \to +\infty} \hat{\theta}_k = \lim_{k \to +\infty} \overline{\theta}_k = q_{\mathbf{H}}$.

By (20), taking the limit, we can conclude that

$$\mathsf{E}[Q_{\mathbf{H}} \mid R(I) = n', N(J) = n'', N_{(I)}(J) = m] = q_{\mathbf{H}}. \tag{21}$$

At this point, it is possible to write a formula for the value of the patent accumulated in $\mathbf{H}$, $\widetilde{C}_{\mathbf{H}}$, taking also into account the deterministic renewal process $\{(T_i, X_i)\}_{i \in \mathrm{N}}$. Starting from $q_{\mathbf{H}}$ given in (21) we subtract the deterministic total amount of the annuities describing the periodic patent renewal process, under the constraints formalized in property Π_t and exit time τ^*. In order to formalize the intervention of property Π_t, the summation of the fees are stopped at the index $j^* - 1$, according to formula (6). We also stress that an expected value must be computed as the presence of the stochastic threshold $\tau^?$ maintains randomness in our framework.

$$\widetilde{C}_{\mathbf{H}} = q_{\mathbf{H}} - \mathsf{E}\Big[\sum_{j=1}^{j^*-1} X_j \cdot \mathbf{1}_{\{T_j \in \mathbf{H} \cap [0, \phi_{|\tau_i}(\tau^*))\}}\Big] =$$

$$= q_{\mathbf{H}} - \sum_{j=1}^{j^*-1} X_j \mathsf{E}[\mathbf{1}_{\{T_j \in \mathbf{H} \cap [0, \phi_{|\tau_i}(\tau^*))\}}] =$$

$$q_{\mathbf{H}} - \sum_{j=1}^{j^*-1} X_j P[T_j \in \mathbf{H} \cap [0, \phi_{|\tau_i}(\tau^*))] =$$

$$= q_{\mathbf{H}} - \sum_{j=1}^{j^*-1} X_j \cdot \mathbf{1}_{\{T_j \in \mathbf{H}\}} \cdot P(T_j \le \phi_{|\tau_i}(\tau^*)) =$$

$$= q_{\mathbf{H}} - \sum_{j=1}^{j^*-1} X_j \cdot \mathbf{1}_{\{T_j \in \mathbf{H}\}} \cdot P(T_j \le \lambda^*) =$$

$$= q_{\mathbf{H}} - \sum_{j=1}^{j^*-1} X_j \cdot \mathbf{1}_{\{T_j \in \mathbf{H}\}} \cdot [1 - F_{\lambda^*}(T_j)], \tag{22}$$

where λ^* is the delay of the exit time τ^*, according to formula (12) and F_{λ^*} is the (marginal) distribution function of λ^*.

Equation (22) provides the value of the patent over a fixed time interval, $\mathbf{H}$, under the provision that both the underlying process has never hit the barrier zero before and that, according to (6), the statutory payments have never been greater than the time varying threshold $\gamma(t)$. Note that, even without assuming no expiration date, the result is given in a closed form, instead of being solved numerically as a partial differential equation. Not surprisingly the patent value is given by a finite value $q_{\mathbf{H}}$ minus the sum of the renewal fees paid till that moment. Despite its intuitive significance, (22) reveals some critical considerations.

Recall that $q_{\mathbf{H}}$ is a theoretical computation of the total size of the jumps in the patent value over a given time interval $\mathbf{H}$, due to the jumps in the underlying security, conditional on past information. This quantity is the algebraic sum of three components:

$$q_{\mathbf{H}} = q_{\mathbf{H}}^{(a)} + q_{\mathbf{H}}^{(b)} - q_{\mathbf{H}}^{(c)}, \tag{23}$$

where $q_{\mathbf{H}}^{(a)}$ is given by the jumps in the patent due to the jumps in the underlying security occurred in $\mathbf{H}$; $q_{\mathbf{H}}^{(b)}$ represents the jumps in the patent value due to the jumps in the underlying security occurred before $\mathbf{H}$ and propagated in $\mathbf{H}$; finally, $q_{\mathbf{H}}^{(c)}$ represents the jumps in the patent due to the jumps in the underlying security occurred in $\mathbf{H}$ and propagated after $\mathbf{H}$.

For a given renewal schedule, formula (23) is useful to compare the patent value obtained under (22) with the patent value obtained under an alternative naive model. By naive model, we mean a model in which no delay between the underlying security and the patent jump is assumed. Under this restrictive hypothesis, let us denote by $\tilde{q}_{\mathbf{H}}$ the total size of the jumps in the patent value over a given time interval $\mathbf{H}$, due to the jumps in the underlying security. The no delay condition implies that

$$\tilde{q}_{\mathbf{H}} = q_{\mathbf{H}}^{(a)}, \tag{24}$$

where $q_{\mathbf{H}}^{(a)}$ is defined as in (23).

By (23) and (21), we generally have that $q_{\mathbf{H}} \neq \tilde{q}_{\mathbf{H}}$, since there is no reason to expect $q_{\mathbf{H}}^{(b)} = q_{\mathbf{H}}^{(c)}$. Therefore, we argue that paying a little price in terms of algebraic effort, the model can better fit the real world, providing us with the patent value in a closed form solution. More remarkably, (22) and (23) show potential source of bias when estimating the value of patents in the "no delay" hypothesis. So far, the comparison between our model and an alternative naive one has been kept as simple as possible, but the bias can be worsened if one considers also that naive models contemplate neither the possibility of positive jumps nor the decay process of the transmission mechanism. Furthermore, if a naive model is applied to determine the value of patent rights in two countries or in two sectors, there is no reason to believe that the two biases will take on the same sign and the same intensity, hence the comparison can be meaningless.

4 THE ENDOGENOUS RENEWAL THRESHOLD

In this section, the deterministic time varying renewal threshold, $\gamma(t)$ is avoided to be determined at endogenous level. In so doing we can show more clearly the strict (apparent) similarity between the results of this model and the results obtained in other works under simpler hypotheses. This comparison is interesting in that it shows how it is possible to use SMPPs consistently with the findings of the literature on patent renewals. At the same time, it highlights the key differences concealed in the seemingly similar results.

The decision of not renewing the patent is irreversible in the sense that once the fee is not paid for the first time the patent dies out forever, and there is no possibility to restart it again in the future, should the net gain turns positive. If, at each renewal time, the holder compares X_j to the expected gain accruing over the next time interval, $[T_j; T_{j+1})$, he does not properly take into account the irreversibility of the choice, making a myopic decision. In order to avoid this trivial mistake, all future fees should be properly taken into account and compared to the whole set of future expected gains.

Therefore, a more consistent way to endogeneize the renewal threshold $\gamma(t)$ is given by considering the discounted revenues as well as the total amount of the discounted future fees. We provide an estimate of the discounted patent value at a fixed date, say $\bar{t} > 0$, on the basis of the information collected in a previous period of time. Two main terms feature in the valuation mechanism: the first driven by the jumps in the underlying security, the second concerning the fees. The introduction of the catastrophic event due to τ^* should also be considered, since it goes beyond the patent holder's control. Inversely, the constraint on the discounted fees, formalized in property Π_t, has to be removed.

As already said, we need to compute the discounted gain from the patent after $\bar{t}$. Keeping the theoretical framework and the notation unchanged, without loss of generality, we can assume that $\bar{t} \notin \mathbf{I}$, i.e. $\bar{t} > T' + s$. Moreover, we introduce a discount factor $\beta \in (0,1)$. Consider a partition of H:

$$\Psi_h := \{G_v^{(h)}\}_{v=1,\dots,h}, \quad h \in \mathcal{N}, \tag{25}$$

where $G_v^{(h)} := (t_{v-1}^{(h)}, t_v^{(h)}] \times [-b,b]$, with $v = 1, ?, h$, $t_0^{(h)} = \bar{t}$, $t_h^{(h)} = T$ and, for each h, $\{t_v^{(h)}\}$ is increasing with respect to v.

A further refined partition of H can be obtained by the intersection of the partitions defined in (18) and (25). We have

$$\Delta_k \cap \Psi_h = \left\{H_s^{(k)} \cap G_v^{(h)}\right\}_{v \le h;\ s \le k} \quad k,\, h \in \mathcal{N}, \tag{26}$$

Fix the four integers s, v, h, k. Analogously to formula (19), we denote by $b_{s,v}^{(k,h)}$ the expected number of jumps in the patent value observed in the time interval $(t_{v-1}^{(h)}, t_v^{(h)}]$ with size $(c_{s-1}^{(k)}, c_s^{(k)}]$, conditioned on the previous history in period $\mathbf{I}$, i.e.

$$b_{s,v}^{(k,h)} \equiv \mathsf{E}\left[N(H_s^{(k)} \cap G_v^{(h)}) \mid R(I) = n', N(J) = n'', N_{(I)}(J) = m\right]. \tag{27}$$

In this case, Proposition 2 can be rewritten in order to compute $b_{s,v}^{(k,h)}$, for any $(k,h) \in \mathcal{N}^2$ and $(s,v) \in \{1,\dots,k\} \times \{1,\dots,h\}$.

Proposition 3

$$b_{s,v}^{(k,h)} = \sum_{n=0}^{+\infty}\sum_{l=0}^{n} n \frac{[M^*_{(\bar{I})}(H_s^{(k)} \cap G_v^{(h)})]^{n-l}}{(n-l)!} \frac{[M^*_{(I)}(H_s^{(k)} \cap G_v^{(h)})]^{l}}{l!} \cdot$$

$$\cdot \int_0^{+\infty} \lambda^{n-l} e^{-\lambda M^*_{(\bar{I})}(H_s^{(k)} \cap G_v^{(h)})} u(\lambda; I, J, n', n'', m)\mathbf{d}\lambda \cdot \int_0^{+\infty} \lambda^{l} e^{-\lambda M^*_{(I)}(H_s^{(k)} \cap G_v^{(h)})} u(\lambda; I, J, n', n'', m)\mathbf{d}\lambda,$$

where $u(\lambda; I, J, n', n'', m)$ is defined as in Proposition 2.

We define as $Q_{\bar{t}}$ the discounted expected value of the patent at time $\bar{t}$, the value of which uniquely stems from the jumps in the dynamics of the underlying process. Proposition 3 allows to get an upper and a lower approximation of $\mathsf{E}[Q_{\bar{t}} \mid R(I) = n', N(J) = n'', N_{(I)}(J) = m]$.

Consider the following sequences:

$$\overline{\phi}_{k,h} := \sum_{s=1}^{k}\sum_{v=1}^{h} c_s^{(k)} \beta^{t_{v-1}^{(h)} - \bar{t}} b_{s,v}^{(k,h)},$$

$$\hat{\phi}_{k,h} := \sum_{s=1}^{k}\sum_{v=1}^{h} c_{s-1}^{(k)} \beta^{t_{v}^{(h)} - \bar{t}} b_{s,v}^{(k,h)}.$$

For any $h, k \in \mathsf{N}$, we have

$$\hat{\phi}_{k,h} \leq \mathsf{E}[Q_{\bar{t}} \mid R(I) = n', N(J) = n'', N_{(I)}(J) = m] \leq \overline{\phi}_{k,h}. \tag{28}$$

$\{\hat{\phi}_{k,h}\}$ is non-decreasing and $\{\overline{\phi}_{k,h}\}$ is non-increasing with respect to k and h. Moreover, there exists a nonnegative constant dependent on $\bar{t}$, named $p(\bar{t})$, such that $\lim_{k,h \to +\infty} \hat{\phi}_{k,h} = \lim_{k,h \to +\infty} \overline{\phi}_{k,h} = p(\bar{t})$.

By (20), taking the limit, we can conclude that

$$\mathsf{E}[Q_{\bar{t}} \mid R(I) = n', N(J) = n'', N_{(I)}(J) = m] = p(\bar{t}). \tag{29}$$

Denote as $\widetilde{C}_{\bar{t}}$ the discounted patent value valuated at time $\bar{t}$, which is drawn taking into consideration the patent renewal process $\{(T_i, X_i)\}_{i \in \mathcal{N}}$. Proceeding in agreement with the previous section and taking into account the random exit time τ^*, $\widetilde{C}_{\bar{t}}$ can be computed as:

$$\widetilde{C}_{\bar{t}} = p(\bar{t}) - \mathsf{E}\Big[\sum_{j=1}^{+\infty} X_j \beta^{T_j - \bar{t}} \cdot \mathbf{1}_{\{T_j \in [\bar{t}, \phi_{|\tau_i}(\tau^*))\}}\Big] =$$

$$= p(\bar{t}) - \sum_{j=1}^{+\infty} X_j \beta^{T_j - \bar{t}} \cdot \mathbf{1}_{\{T_j \in [\bar{t}, T]\}} \cdot P(T_j \le \phi_{|\tau_i}(\tau^*)) =$$

$$= p(\bar{t}) - \sum_{j=1}^{+\infty} X_j \beta^{T_j - \bar{t}} \cdot \mathbf{1}_{\{T_j \in [\bar{t}, T]\}} \cdot P(T_j \le \lambda^*) =$$

$$= p(\bar{t}) - \sum_{j=1}^{+\infty} X_j \beta^{T_j - \bar{t}} \cdot \mathbf{1}_{\{T_j \in [\bar{t}, T]\}} \cdot [1 - F_{\lambda^*}(T_j)]. \quad (30)$$

The renewal threshold is so endogeneized by posing $\tilde{C}_{\bar{t}} = \gamma(\bar{t}), \quad \forall\ \bar{t}.$

Equation (30) is perfectly in line with the literature on renewals (e.g. Shankerman and Pakes (1986) eq. (1), Pakes and Simpson (1989) eq. (1), Baudry and Dumont (2006) eq.(4)), in that the value of the patent at time $\bar{t}$ is nothing but the expected net gain discounted from the valuation time, $\bar{t}$, till its possible death. This value is given by the difference of two components. A first component stemming from the discounted expected value of the jumps in the underlying security, $p(\bar{t})$, decreased by the second one, the discounted value of the total amount of the fees payable in the future. In this quantity, the term $[1 - F_{\lambda^*}(T_j)]$ captures the presence of the exit time τ^* associated to a catastrophic (negative) jump of the underlying security, that is responsible for the collapse of the patent value. That is, if a renewal fee X_j is due in a time T_j smaller than the delay λ^* of the stochastic time τ^*, it has to be inserted in the computation. Otherwise X_j does not feed into the summation in (30).

In spite of the close similarity to the aforementioned literature, the crucial point that causes us to depart from the $p(\bar{t})$ term which plays the analogous role of $q_{\mathbf{H}}$ in (22). The endogenization of the renewal threshold $\gamma(t)$ does not alter the reasoning put forward commenting on (22), which still applies. But the appearance of (30) makes clearer that our set up is a generalization of the existing literature, aimed at providing actual features that can induce a non negligible difference in the value of patents. Defining by C the value of the patent under an alternative model, the reasons that make $\tilde{C}_{\bar{t}}$ differ from C - such as positive jumps and delay and decay in the jump transmission mechanism - still hold in (30). More specifically, $\tilde{C}_{\bar{t}} = C$ is a special case of (30) and its occurrence is purely accidental.

Again, to a quantitative appraisal of the difference is far beyond the scope of this paper. We limit ourselves to bring it to notice.

5 CONCLUSIONS

This paper is a generalization of the existing models for valuing a patent in a real options framework. Briefly speaking, the generalization consists in accounting for positive and negative random jumps, inducing possible delays in the transmission mechanism between the jumps in the underlying process and the corresponding jumps in the patent value, and considering a decay process of the intensity of the shocks occurred to the underlying state. If a shock occurs in the underlying process when the patent is close to expiration, *caeteris paribus*, it will affect the patent with lower intensity.

We have suggested that the inclusion of these hypotheses is essential to add realism to the valuation process. In turn, this step also brings significant improvements, such as a solution to patent valuation in a closed form, without assuming the absence of expiration and considering the presence of non killing jumps. It follows that patent evaluation can greatly differ whether our model is adopted or a more naive approach is followed. Many factors account for that difference. As already said above, and according to some empirical works, the number of non deadly jumps seems to be as remarkable as the presumable cumulated change in the high-valued patent. The delay in the transmission mechanism is such that the patents value is accidentally equal under the two models, and the differences in both sign and magnitude can be hardly predicted.

Giving an estimate of the value of patent rights through a naive model, from a policy point of view, can lead to biased results that hinder a meaningful comparison between sectors and countries. If policy recommendations and interventions are based on a biased comparison, misleading incentives may be put into play.

Form a theoretical point of view, and as a possible development for further research, it would be interesting to apply this set up to a patent race context, addressing the question of what may happen to the value of waiting to invest when the transmission delay and the other above-mentioned features are concerned. On one hand, the delay increases the value of waiting before incurring the sunk cost of patenting a new invention. If a positive jump in the underlying process has not fully propagated to the derivative, the trigger value of the investment will be successively reached, making still worth waiting. On the other hand, the possibility of positive jumps makes the would-be patentee more eager to take out the patent to enjoy the benefits. Yet, the presence of the delay makes the "winner takes all" hypothesis not sustainable, at least before the delay is sufficiently disappeared. Does the removal of this hypothesis lessen the competition and therefore the result of the firms race? What consequences there might be in terms of R&D? Another key point that would be worthwhile tackling in a succeeding study is the empirical evaluation of patent rights under the two different approaches, thorough an econometric estimate or a simulation approach.

APPENDIX: SOME THEORY ON SPATIAL MIXED POISSON PROCESSES

This Appendix contains the definition and some recent results on SMPPs, that can be already found in the literature.

Let us define a probability space $(\Omega, \mathbf{F}, P)$ containing all the random variables involved in our discussion. SMPPs are particular spatial point processes. For a survey of the general theory of spatial point processes, we remind to Daley and Vere-Jones (1988) and Stoyan et al. (1995). Let us consider a measure space $(\mathrm{R}^k, \mathbf{B}(\mathrm{R}^k), M)$, where $\mathbf{B}(\mathrm{R}^k)$ is the Borel σ-algebra and *M* is absolutely continuous with respect to the Lebesgue measure. We also introduce a nonnegative random variable Λ with probability distribution $U : \mathrm{R} \to [0,1]$.

Definition 1

A spatial process R is Mixed Poisson with mixing distribution U and baseline intensity measure M(•) if and only if, for $I \in \mathbf{B}(\mathrm{R}^k)$ and for $n \in \mathrm{N}$,

$$P(R(I) = n) = \int_0^\infty e^{-\lambda M(I)} \frac{[\lambda M(I)]^n}{n!} dU(\lambda). \qquad (31)$$

We need some properties on the SMPPs, that have been used in ourwork. Since M is absolutely continuous with respect to the Lebesgue measure, the following result states immediately.

Lemma 1 *A SMPP is a simple point process.*

Now, consider a spatial point process $R \equiv \{X_i\}_{i \in \mathrm{N}}$, with $X_i \in \mathbf{X} \subseteq R^k$, $k \in \mathrm{N}$, for any $i \in R$.

Consider a sequence of i.i.d. random variables $\mathbf{W} = \{W_i\}_{i \in \mathrm{N}}$ taking values on a set $\mathbf{W} \subseteq \mathrm{R}^n$ for some $n \in \mathrm{N}$. Define a transformation $\phi : \mathbf{X} \times \mathbf{W} \to \mathbf{Y} \subseteq \mathrm{R}^k$, where $\phi(\cdot, w) : \mathbf{X} \to \mathbf{Y}$ is measurable and one-to-one foranyfixed $w \in \mathbf{W}$. We define the transformed spatial point process as follows:

$$N = \{\phi(X_i, W_i)\}_{i \in \mathcal{N}} \qquad (32)$$

We can also write $N = \Phi_\phi(R, \mathbf{W})$ instead of (32).

The following theorem is already known in the literature (see e.g. Cinlar (1995)) and a proof based on stochastic geometrical arguments can be found in Foschi and Spizzichino (2008).

Theorem 1. Let R be a SMPP with mixing distribution U and baseline intensity measure M. Consider a sequence of i.i.d. random variables $\mathbf{W} = \{W_i\}_{i\in\mathcal{N}}$, with distribution G and independent of R.

Then $N = \Phi_\phi(R, \mathbf{W})$ is a SMPP with the same mixing distribution U and intensity measure $M^*(J) = \int_{\mathrm{R}^n} M(\phi_w^{-1}(J))dG(w),$ where $J \subseteq \mathbf{Y}$ and $X \in \phi_w^{-1}(J)$ if and only if $\phi(X, w) \in J$.

Theorem 1 is a key result in our work, since it explains the invariance of SMPPs with respect to a very general class of transformations. A further result can be provided concerning conditional estimates of SMPPs realizations. Let us consider three regions $I \subseteq \mathbf{X}, J, H \subseteq \mathbf{Y}$. We aim at estimating the number of points of N fallen in H, knowing the restriction of the processes *R* and *N* to the regions *I* and *J* respectively. First of all, we need to introduce a notation describing the number of points of R fallen in *I* and sent by the transformation ϕ into J or into *H*. The latter quantities are represented by the random variables $N_{(I)}(J)$ and $N_{(I)}(H)$ respectively, given by:

$$N_{(I)}(K) \equiv \sum_{\alpha\in A} \mathbf{1}_{\{\phi(X_\alpha, W_\alpha)\in K\}} \mathbf{1}_{\{X_\alpha \in I\}}, \quad K \in \{J, H\}. \tag{33}$$

A straightforward computation proves the following result:

Lemma 2. For a given $I \subseteq \mathbf{X}$, $N_{(I)}$ can be thought of as a SMPP of its own, with mixing distribution U and baseline intensity measure

$$M^*_{(I)}(J) = \int_{\mathcal{R}^n} M(I \cap \phi_w^{-1}(J))dG(w)$$

Define the event:

$$E_{(l,m)} \equiv \{N_{(I)}(H) = l, N_{(I)}(J) = m\}, \quad l, m \in R \cup \{0\}$$

In Cerqueti et al. (2009), the following theorem is proved:

Theorem 2. *For arbitrary subsets* I, J, H, *we have*

$$P(N(H) = n \mid R(I) = n', N(J) = n'', E_{(l,m)}) \tag{34}$$

$$= \int_0^\infty \frac{[\lambda M^*_{(\bar{I})}(H)]^{n-l}}{(n-l)!} e^{-\lambda M^*_{(\bar{I})}(H)} u(\lambda; I, J, n', n'', m) d\lambda,$$

where

$$u(\lambda; I, J, n', n'', m) = \frac{\lambda^{n''-m+n'} e^{-\lambda[M^*_{(\bar{I})}(J)+M(I)]} u(\lambda)}{\int_0^\infty \lambda^{n''-m+n'} e^{-\lambda[M^*_{(\bar{I})}(J)+M(I)]} u(\lambda) d\lambda} \tag{35}$$

Theorem 2 also provides an estimate of the parameter Λ.We notice, in fact, that $u(\lambda; I, J, n', n'', m)$ coincides with the posterior distribution of Λ given the observation of the event $\{R(I) = n', N(J) = n'', N_{(I)}(J) = m\}$,i.e. $u(\lambda; I, J, n', n'', m) = U(\lambda \mid R(I) = n', N(J) = n'', N_{(I)}(J) = m)$ .This fact allows us to explicitly derive the distribution of the transformed process N , starting from the definition of the SMPP R .

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